Efficient Coalgebraic Partition Refinement

Thorsten Wißmann

Joint work with:
Ulrich Dorsch, Stefan Milius, Lutz Schröder

Friedrich-Alexander-Universität Erlangen-Nürnberg

CONCUR 2017
September 7, 2017

Efficient Coalgebraic Partition Refinement

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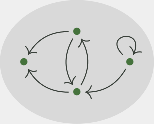
1. Coalgebras:

State based systems



Labels, Non-Determinism,
Probabilities, Automata,
... and their combinations!

Efficient Coalgebraic Partition Refinement

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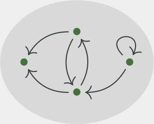

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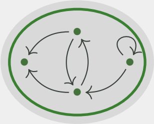
2. Partition Refinement:

Successively distinguish different behaviour



Efficient Coalgebraic Partition Refinement

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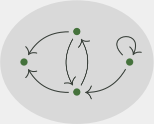

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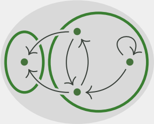

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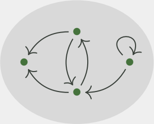

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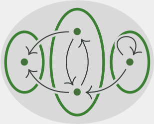


Efficient Coalgebraic Partition Refinement

3. Efficiency:

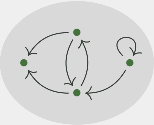
(a) Incrementally compute partitions

(b) Complexity Analysis

$$\mathcal{O}(\underset{\underset{\text{Edges}}{\uparrow}}{m} \cdot \log \underset{\underset{\text{States}}{\uparrow}}{n})$$

1. Coalgebras:

State based systems



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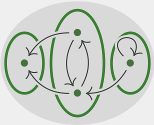

Share Common Structure & Ideas

Similar Run-Time

Variations in Details

Share Common
Structure & Ideas

Deterministic
Finite Automata

$n \cdot \log n$	$\lvert A\rvert \cdot n \cdot \log n$
Hopcroft '71	Gries '73 Knuutila '01

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Transition Systems

$m \cdot \log n$
Paige, Tarjan '87

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Segala Systems

$m \cdot n \cdot (\log m + \log n)$
Baier, Engelen, Majster-Cederbaum '00

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Labelled Transition Systems

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Valmari '09

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Weighted Systems "Markov Chain Lumping"

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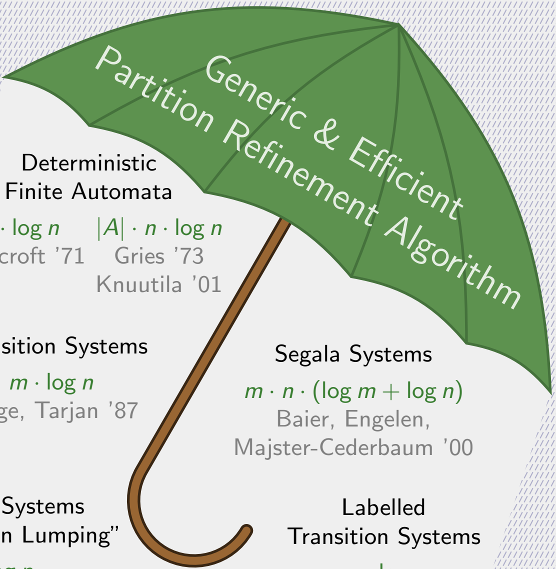

Generic & Efficient
Partition Refinement Algorithm

Deterministic
Finite Automata

$n \cdot \log n$ Hopcroft '71

$|A| \cdot n \cdot \log n$ Gries '73, Knuutila '01

Transition Systems

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Weighted Systems
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Ingredient 1: Factorizations

Equivalence Relations $\cong$ Quotients $\cong$ Partitions

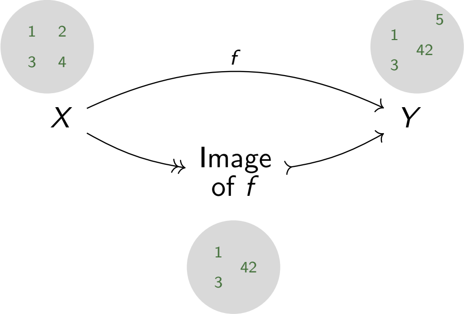
Kernels $\cong$ Regular Epimorphisms

Ingredient 1: Factorizations

Equivalence Relations ≅ Quotients ≅ Partitions

Kernels ≅ Regular Epimorphisms

Category with (Regular Epi,Mono)-Factorizations



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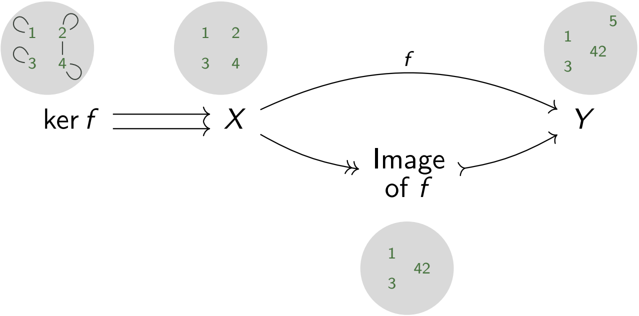

$$\begin{tikzcd}
\ker f \arrow[r, shift left] \arrow[r, shift right] & X \arrow[rr, bend left, "f"] \arrow[dr, two heads] & & Y \\
& & \text{Image of } f \arrow[ur, tail] &
\end{tikzcd}$$

$$\ker f = \{(x_1, x_2) \in X^2 \mid f(x_1) = f(x_2)\}$$

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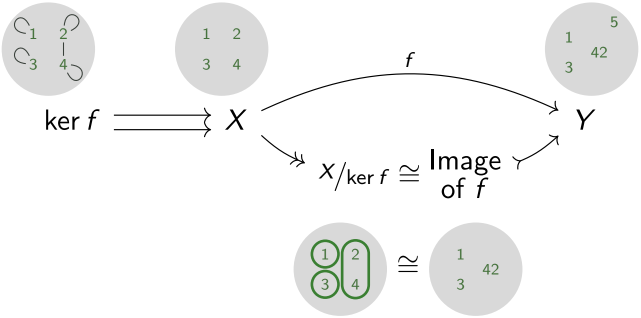

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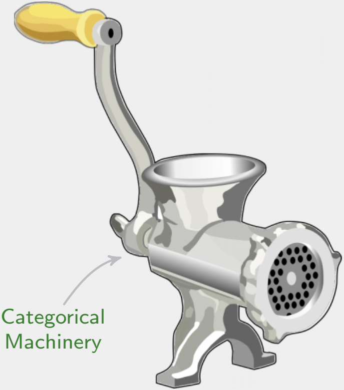
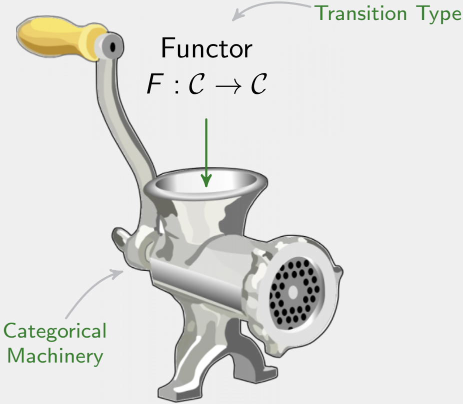
Ingredient 2: Coalgebra – Generic state based systems



Categorical Machinery

Ingredient 2: Coalgebra – Generic state based systems



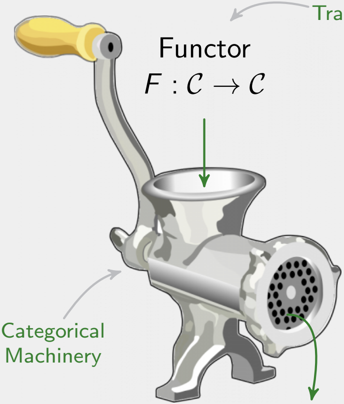
Transition Type

Functor

$F : \mathcal{C} \to \mathcal{C}$

Categorical Machinery

Ingredient 2: Coalgebra – Generic state based systems



Transition Type

Functor

$F : \mathcal{C} \to \mathcal{C}$

Categorical Machinery

F-Coalgebra

$C \xrightarrow{c} FC$

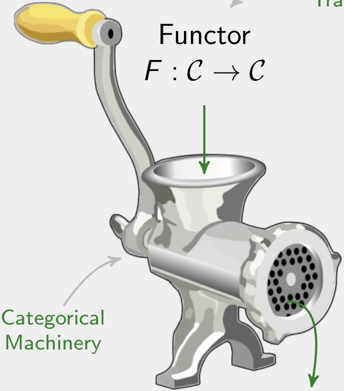
Ingredient 2: Coalgebra – Generic state based systems

Powerset $\mathcal{P}$

Transition Type

Functor

$F : \mathcal{C} \to \mathcal{C}$



Categorical Machinery

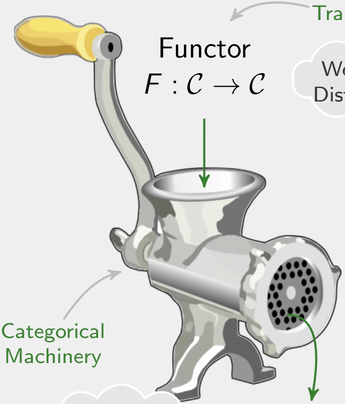
F-Coalgebra

$C \xrightarrow{c} FC$

Transition System

$C \xrightarrow{c} \mathcal{P}C$

Ingredient 2: Coalgebra – Generic state based systems



Powerset $\mathcal{P}$

Transition Type

Functor

$F : \mathcal{C} \to \mathcal{C}$

Weights $\mathbb{R}^{(-)}$
Distributions $\mathcal{D}$

Categorical Machinery

Markov Chain
$C \xrightarrow{c} \mathbb{R}^C$

Transition System
$C \xrightarrow{c} \mathcal{P}C$

F-Coalgebra

$C \xrightarrow{c} FC$

Ingredient 2: Coalgebra – Generic state based systems

Powerset $\mathcal{P}$

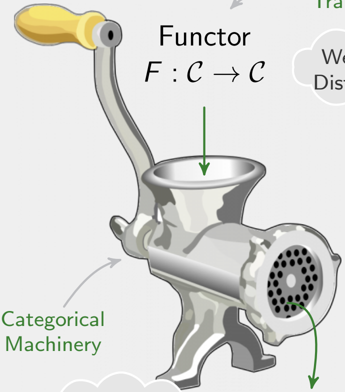
Transition Type

Functor

$F : \mathcal{C} \to \mathcal{C}$

Weights $\mathbb{R}^{(-)}$
Distributions $\mathcal{D}$

Automata
$2 \times (-)^A$



Categorical Machinery

F-Coalgebra

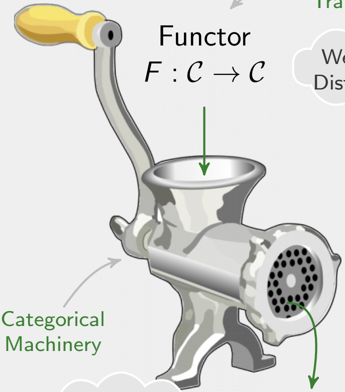
$C \xrightarrow{c} FC$

Markov Chain
$C \xrightarrow{c} \mathbb{R}^C$

Transition System
$C \xrightarrow{c} \mathcal{P}C$

DFA
$C \xrightarrow{c} 2 \times C^A$

Ingredient 2: Coalgebra – Generic state based systems



Functor

$F : \mathcal{C} \to \mathcal{C}$

Transition Type

Powerset $\mathcal{P}$

Simple Segala $\mathcal{P}(A \times -) \cdot \mathcal{D}$

Weights $\mathbb{R}^{(-)}$
Distributions $\mathcal{D}$

Automata $2 \times (-)^A$

Categorical Machinery

F-Coalgebra

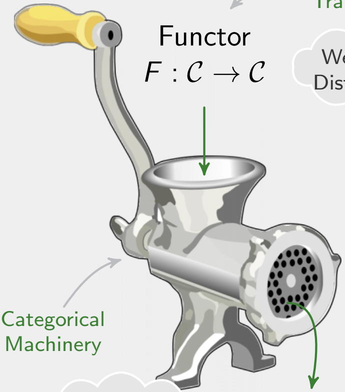
$C \xrightarrow{c} FC$

Markov Chain $C \xrightarrow{c} \mathbb{R}^C$

Transition System $C \xrightarrow{c} \mathcal{P}C$

DFA $C \xrightarrow{c} 2 \times C^A$

Ingredient 2: Coalgebra – Generic state based systems



Transition Type

Functor

$F : \mathcal{C} \to \mathcal{C}$

Powerset $\mathcal{P}$

Weights $\mathbb{R}^{(-)}$
Distributions $\mathcal{D}$

Automata
$2 \times (-)^A$

Simple Segala
$\mathcal{P}(A \times -) \cdot \mathcal{D}$

General Segala
$\mathcal{P} \cdot \mathcal{D}(A \times -)$

Categorical
Machinery

F-Coalgebra

$C \xrightarrow{c} FC$

Markov Chain
$C \xrightarrow{c} \mathbb{R}^C$

Transition System
$C \xrightarrow{c} \mathcal{P}C$

DFA
$C \xrightarrow{c} 2 \times C^A$

Ingredient 2: Coalgebra – Generic state based systems

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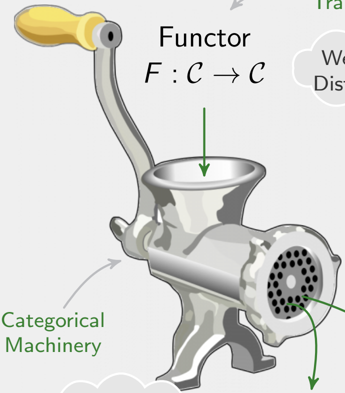


Categorical Machinery

F-Coalgebra

$C \xrightarrow{c} FC$

Markov Chain

$C \xrightarrow{c} \mathbb{R}^C$

Transition System

$C \xrightarrow{c} \mathcal{P}C$

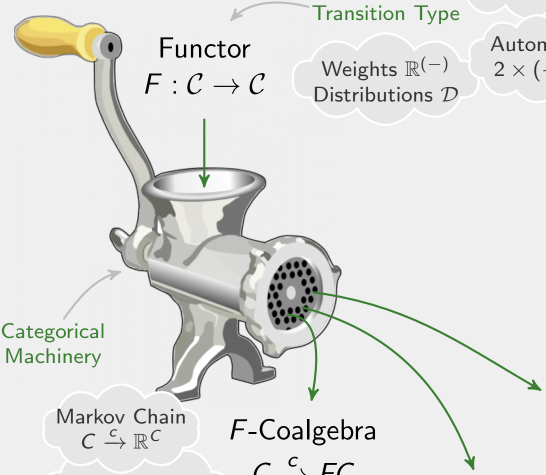
DFA

$C \xrightarrow{c} 2 \times C^A$

Homomorphism

$C \xrightarrow{h} C'$

Ingredient 2: Coalgebra – Generic state based systems



Transition Type

Functor

$F : \mathcal{C} \to \mathcal{C}$

Powerset $\mathcal{P}$

Weights $\mathbb{R}^{(-)}$
Distributions $\mathcal{D}$

Automata
$2 \times (-)^A$

Simple Segala
$\mathcal{P}(A \times -) \cdot \mathcal{D}$

General Segala
$\mathcal{P} \cdot \mathcal{D}(A \times -)$

Categorical Machinery

F-Coalgebra

$C \xrightarrow{c} FC$

Markov Chain
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Transition System
$C \xrightarrow{c} \mathcal{P}C$

DFA
$C \xrightarrow{c} 2 \times C^A$

Homomorphism

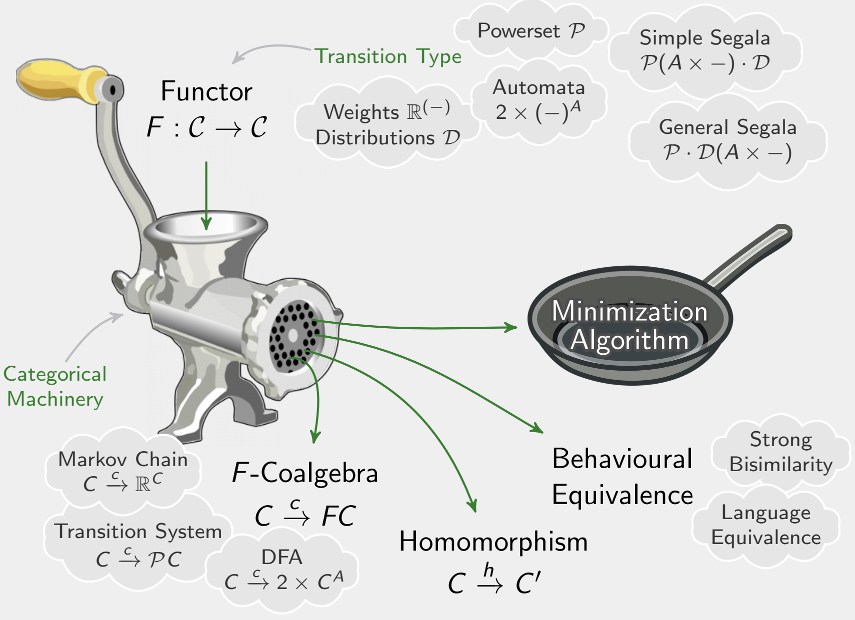
$C \xrightarrow{h} C'$

Behavioural Equivalence

Strong Bisimilarity

Language Equivalence

Ingredient 2: Coalgebra – Generic state based systems



Transition Type

Functor
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Powerset $\mathcal{P}$

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Automata
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Simple Segala
$\mathcal{P}(A \times -) \cdot \mathcal{D}$

General Segala
$\mathcal{P} \cdot \mathcal{D}(A \times -)$

Categorical Machinery

Minimization Algorithm

F-Coalgebra
$C \xrightarrow{c} FC$

Markov Chain
$C \xrightarrow{c} \mathbb{R}^C$

Transition System
$C \xrightarrow{c} \mathcal{P}C$

DFA
$C \xrightarrow{c} 2 \times C^A$

Homomorphism
$C \xrightarrow{h} C'$

Behavioural Equivalence

Strong Bisimilarity

Language Equivalence

The Coalgebraic Task

For a functor $F : \mathcal{C} \to \mathcal{C}$

$$\begin{tikzcd}
\text{Given a coalgebra} & C \arrow[r, "c"] \arrow[d, two heads, "h"'] & FC \arrow[d, two heads, "Fh"] \\
\text{find the simple quotient} & C' \arrow[r, "c'"] & FC'
\end{tikzcd}$$

no proper quotient

all equivalent behaviours identified

The Coalgebraic Task

For a functor $F : \mathcal{C} \to \mathcal{C}$

Given a coalgebra

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Instance

For $2 \times (-)^A : \mathsf{Set} \circlearrowleft$

Automata minimization

The Coalgebraic Task

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Given a coalgebra

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Instance

Instance

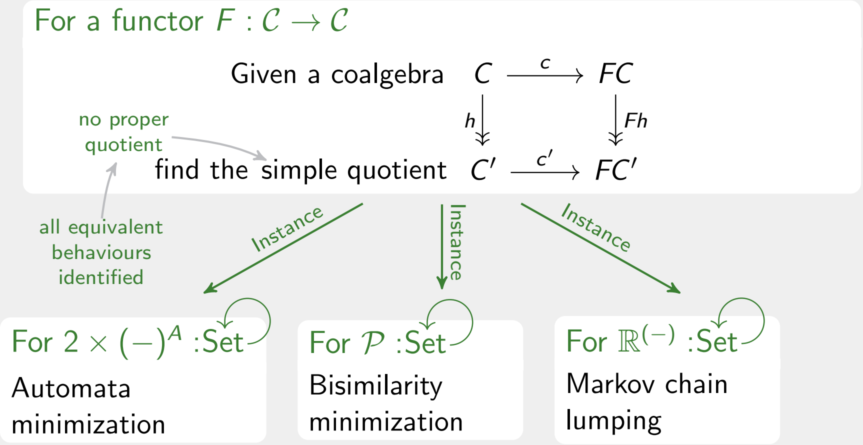
For $2 \times (-)^A$:Set

Automata minimization

For $\mathcal{P}$:Set

Bisimilarity minimization

The Coalgebraic Task



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For $2 \times (-)^A$:Set

Automata minimization

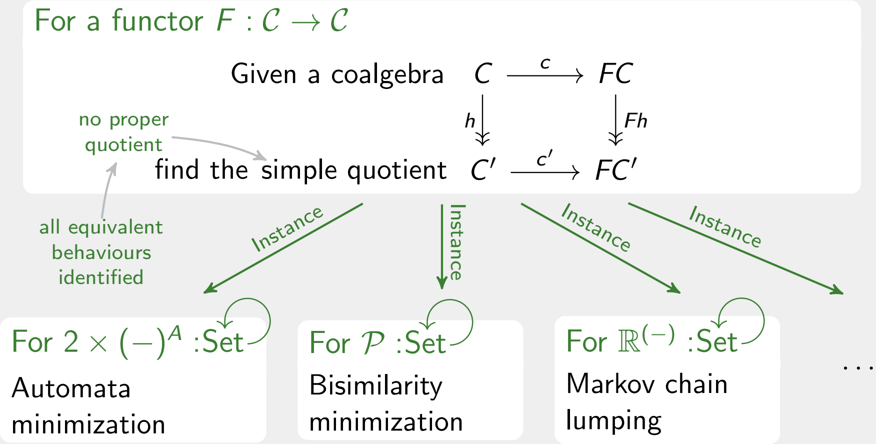
For $\mathcal{P}$:Set

Bisimilarity minimization

For $\mathbb{R}^{(-)}$:Set

Markov chain lumping

The Coalgebraic Task



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For $2 \times (-)^A$:Set

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…

1. Assume everything equivalent

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2. Have a quotient on C

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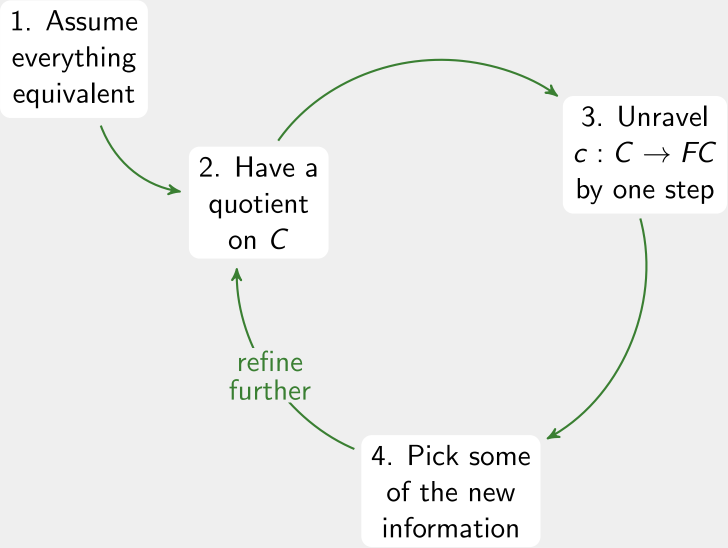
3. Unravel $c : C \to FC$ by one step



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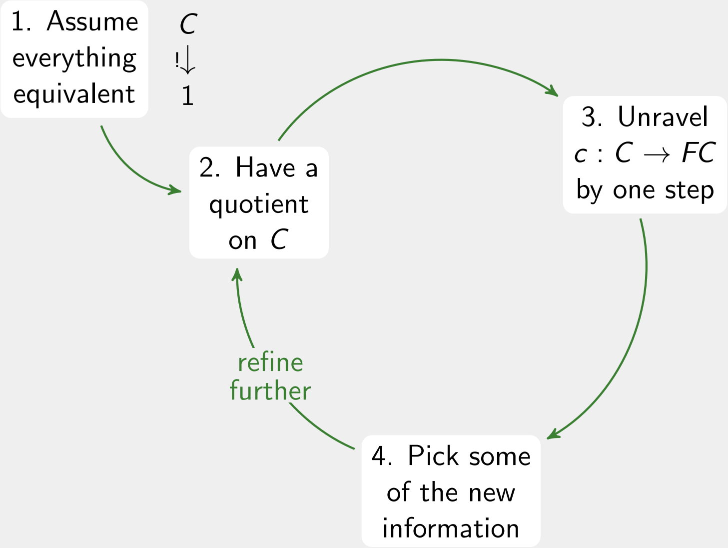
4. Pick some of the new information



1. Assume everything equivalent

$$\begin{tikzcd} C \arrow[d, "!"'] \\ 1 \end{tikzcd}$$

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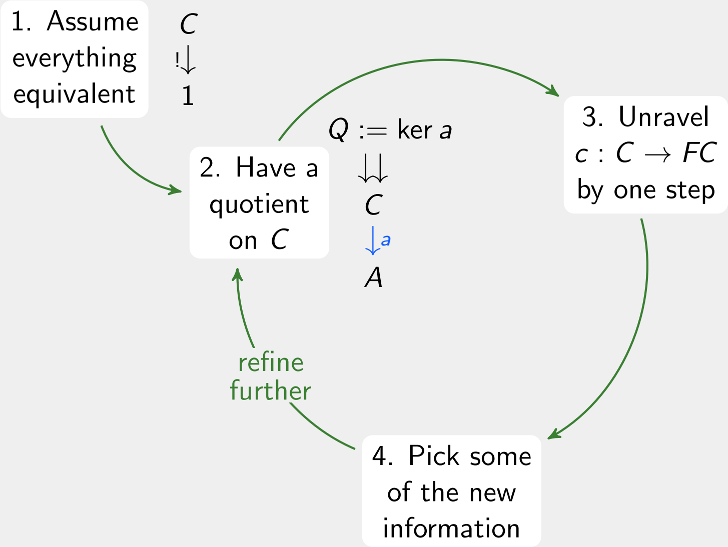
4. Pick some of the new information

refine further



1. Assume everything equivalent

$$\begin{tikzcd} C \arrow[d, "!"'] \\ 1 \end{tikzcd}$$

2. Have a quotient on C

$$\begin{tikzcd} Q := \ker a \arrow[d, shift left] \arrow[d, shift right] \\ C \arrow[d, "a"] \\ A \end{tikzcd}$$

3. Unravel $c : C \to FC$ by one step

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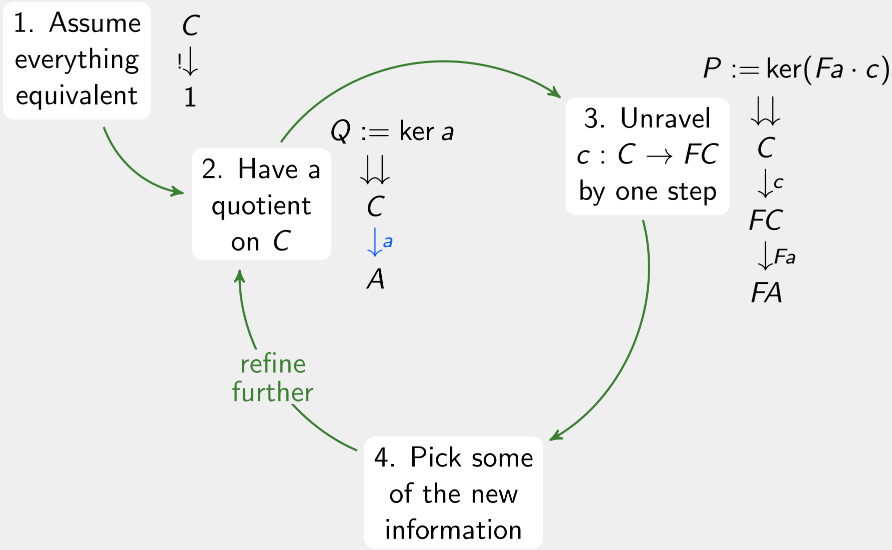
refine further



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$$\begin{tikzcd} P := \ker(Fa \cdot c) \arrow[d, shift left] \arrow[d, shift right] \\ C \arrow[d, "c"] \\ FC \arrow[d, "Fa"] \\ FA \end{tikzcd}$$

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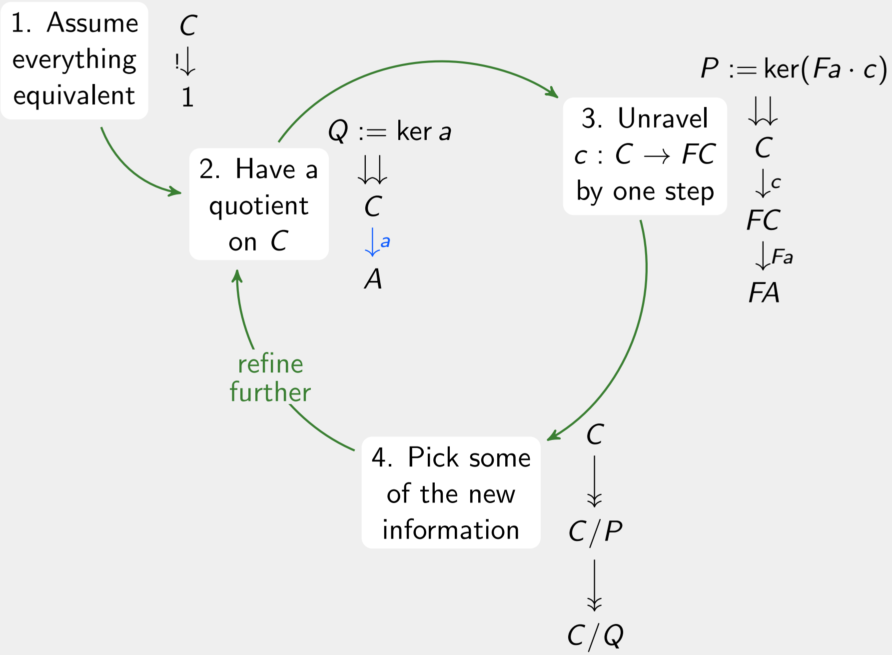

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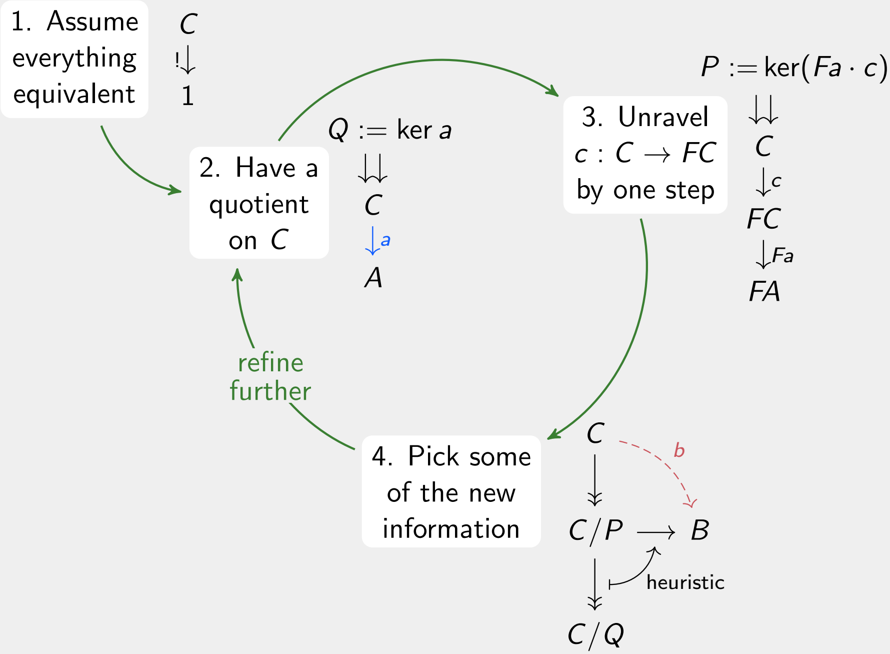
4. Pick some of the new information

$$\begin{tikzcd} C \arrow[d, two heads] \\ C/P \arrow[d, two heads] \\ C/Q \end{tikzcd}$$



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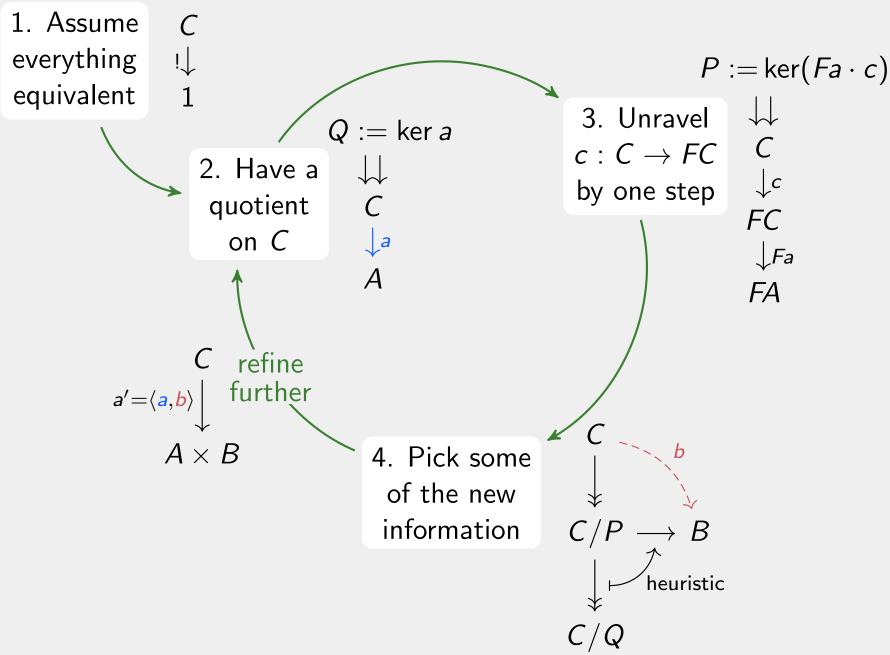
$$\begin{tikzcd} C \arrow[d, two heads] \arrow[dr, dashed, bend left, "b"] & \\ C/P \arrow[r, ""{name=B}] \arrow[d, two heads, ""{name=Q}] & B \\ C/Q & \arrow[from=Q, to=B, mapsto, bend right, "\text{heuristic}"'] \end{tikzcd}$$



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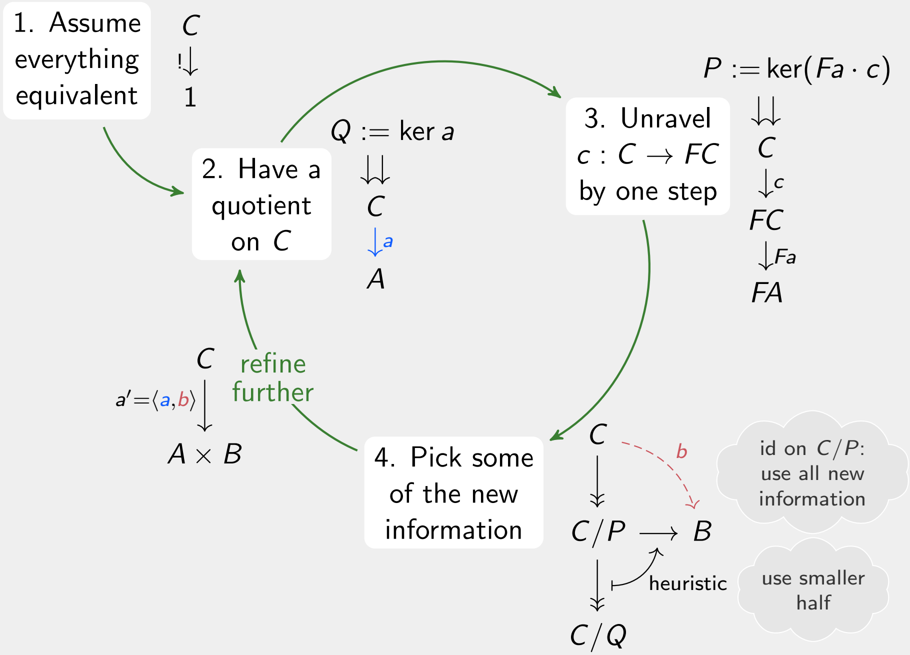
refine further

$$\begin{tikzcd} C \arrow[d, "a'=\langle a,b\rangle"'] \\ A \times B \end{tikzcd}$$



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heuristic

id on C/P: use all new information

use smaller half

refine further

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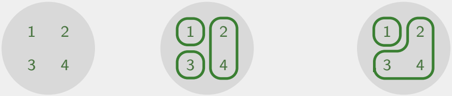
Heuristic



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C \arrow[r, two heads] & C/P \arrow[r, two heads, "p", ""{name=P, below}] \arrow[d, ""{name=D}] & C/Q \\
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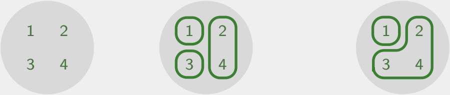

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Use all new information

$B = C/P \rightsquigarrow$ Final Chain algorithm König, Küpper '14

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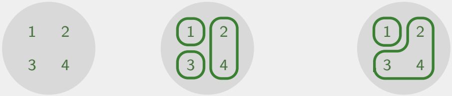

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Use all new information

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Process the smaller half

Surrounding block in C/Q

Let $S \in C/P$, such that $2 \cdot |S| \leq |p(S)|$

$$B = \{\overset{\{3\}}{\text{ChosenBlock}}, \overset{\{2,\,4\}}{\text{SameSurroundingBlock}}, \overset{\{1\}}{\text{RemainingBlocks}}\}$$

Assume

- Finitely complete category $\mathcal{C}$
- (RegularEpi,Mono)-factorisations
- F mono-preserving

Theorem (Correctness)

$$\begin{tikzcd}
C \arrow[r, "c"] \arrow[d, two heads] & FC \arrow[d] \\
C/P_i \arrow[r] & F(C/Q_i)
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If $P_i \cong Q_i$, then this

1. is a coalgebra
2. has no proper quotient

Efficiency: Incremental Partitions

Incremental partitions

$$\begin{tikzcd} Q := \ker a \arrow[d, shift left] \arrow[d, shift right] \\ C \arrow[d, "a"] \\ A \end{tikzcd}$$

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Question: When is $\ker F\langle a, b\rangle = \ker\langle Fa, Fb\rangle$?

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injective and

“zippable”

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Setting for complexity analysis

Category:	Heuristic:	Functor:
Set	smaller half	zippable & **encoding**

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Assumption: Functor encoding

- coalgebra structure as edges with labels

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- Refinement interface { `update()`, `init()`, `weight()` }

⇒ compute "smaller half" intersections in linear time

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Theorem

Overall complexity: $\mathcal{O}((m+n)\cdot \log n)$ for $n = |C|$, $m = \sum_{x \in C} |\flat c(x)|$

System	Functor	Concrete algorithm		Our instantiation
Transition Systems	$\mathcal{P}$	$(m+n)\cdot\log n$ Paige, Tarjan '87	$=$	$(m+n)\cdot\log n$
LTS	$\mathcal{P}(A\times -)$	$(m+n)\cdot\log(m+n)$ Dovier, Piazza, Policriti '04	$=$	$(m+n)\cdot\log(m+n)$
		$(m+n)\cdot\log m$ Valmari '09	$<$	
Markov Chains	$\mathbb{R}^{(-)}$	$(m+n)\cdot\log n$ Valmari, Franceschinis '10	$=$	$(m+n)\cdot\log n$
DFA	$2\times(-)^A$	$n\cdot\log n$ for fixed A, Hopcroft '71	$=$	$n\cdot\log n$
	$2\times\mathcal{P}(A\times -)$	$\lvert A\rvert\cdot n\cdot\log n$, Gries '73, Knuutila '01	$\approx$	$\lvert A\rvert\cdot n\cdot\log n$ $+\lvert A\rvert\cdot n\cdot\log\lvert A\rvert$
Segala Systems	$\mathcal{P}(A\times -)\cdot\mathcal{D}$	$m\cdot n\cdot\log(m\cdot n)$ Baier, Engelen, Majster-Cederbaum'00	$>$	$(m+n)\cdot\log(m+n)$

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Generic & Efficient

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More instances:
further system types
& categories

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Compare to
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Appendix …

Genericity: Initial partiton

Given

$$C \xrightarrow{c} FC$$

Usual partition refinement algorithms

Return coarsest partition compatible with c, refining $C \overset{\kappa}{\twoheadrightarrow} \mathcal{I}$

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$$\Updownarrow$$

Coalgebraic partition refinement for $\mathcal{I} \times F$

For the coalgebra $C \xrightarrow{\langle \kappa, c \rangle} \mathcal{I} \times FC$

Genericity: Composition

If F finitary,

$$C \xrightarrow{c} FG\,C$$

Genericity: Composition

If F finitary,

$$C \xrightarrow{c} FG\,C \qquad \rightsquigarrow \qquad D \xhookrightarrow{d} GC$$

Genericity: Composition

If F finitary,

$$\begin{tikzcd} C \arrow[r, "c"] \arrow[dr, dashed, "c'"'] & FG\,C & & \rightsquigarrow & D \arrow[r, hook, "d"] & GC \\ & FD \arrow[u, "Fd"'] & & & & \end{tikzcd}$$

Genericity: Composition

If F finitary,

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C \arrow[r, "c"] \arrow[dr, dashed, "c'"'] & FG\,C \\
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\end{tikzcd}
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\begin{tikzcd}
D \arrow[r, hook, "d"] & GC
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A coalgebra on Set^2 for the functor $(X, Y) \mapsto (FY, GX)$:

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Examples

$$\mathcal{P} \cdot (A \times (-)) \qquad (2 \times \mathcal{P}) \cdot (A \times (-))$$
$$\mathcal{P} \cdot (A \times (-)) \cdot \mathcal{D} \qquad \mathcal{P} \cdot \mathcal{D} \cdot (A \times (-)) \qquad \ldots$$

Functors F zippable, if

$$F(L+R) \xrightarrow{\text{unzip}} F(L+1) \times F(1+R) \text{ is monic.}$$

E.g. Id, Constants, $\times$, $+$, $\hookrightarrow$, $M^{(-)}$, part. additive

Examples for sets $L = \{a_1, a_2, a_3\}$, $R = \{b_1, b_2\}$, $1 = \{\_\}$

$$a_1\, a_2\, b_1\, a_3\, b_2 \overset{\text{unzip}}{\longmapsto} \begin{array}{l} (a_1 a_2 \,\_\, a_3 \,\_, \\ \quad \_\,\_\, b_1 \,\_\, b_2) \end{array}$$

$(-)^*$ is zippable

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$$\begin{tikzcd} a_1\, a_2\, b_1\, a_3\, b_2 \arrow[d, mapsto, bend left, "\text{unzip}"] \\ (a_1 a_2\, \_\, a_3\, \_,\ \_\, \_\, b_1\, \_\, b_2) \end{tikzcd}$$

$(-)^*$ is zippable

$$\begin{tikzcd} \{a_1, a_2, b_1\} \arrow[d, mapsto, bend left, "\text{unzip}"] \\ (\{a_1, a_2, \_\},\ \{\_, b_1\}) \end{tikzcd}$$

$\mathcal{P}$ is zippable

$$\begin{tikzcd} \{\{a_1, b_1\}, \{a_2, b_2\}\} \arrow[dr, mapsto, "\text{unzip}"'] & & \{\{a_1, b_2\}, \{a_2, b_1\}\} \arrow[dl, mapsto, "\text{unzip}"] \\ & (\{\{a_1, \_\}, \{a_2, \_\}\},\ \{\{\_, b_1\}, \{\_, b_2\}\}) & \end{tikzcd}$$

$\mathcal{P}\mathcal{P}$ is not zippable

~~Composition~~

~~Quotients~~

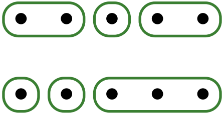
$$A \xleftarrow{a} X \xrightarrow{b} B$$

$\ker a \cup \ker b$ a kernel in Set

$\Leftrightarrow$ $\ker a \cup \ker b$ transitive

$\Leftrightarrow$ $\forall x \in X : [x]_a \subseteq [x]_b$ or $[x]_a \supseteq [x]_b$

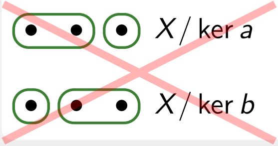
Example



$X/\ker a$

$X/\ker b$

Non-Example



$X/\ker a$

$X/\ker b$

$$A \xleftarrow{a} X \xrightarrow{b} B$$

$\ker a \cup \ker b$ a kernel in Set

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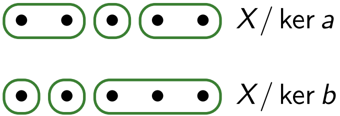
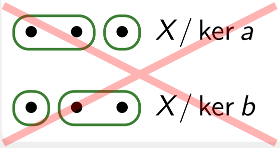
Example



Non-Example



Process smaller half for $X \overset{f}{\twoheadrightarrow} F \overset{g}{\twoheadrightarrow} G$

Find $x \in X$, with $S := [x]_f$, $C := [x]_{gf}$, such that $2 \cdot |S| \leq |C|$.
Return $\langle \chi_S, \chi_C \rangle : X \to 2 \times 2$

Functor encoding

- internal weights W, $w : FX \to \mathcal{P}X \to W$
- edge labels L
- $\flat : FX \to \mathcal{B}_\mathrm{f}(L \times X)$
- $\text{update} : \underbrace{\mathcal{B}_\mathrm{f}(L)}_{\text{edges into } S} \times \underbrace{W}_{\text{weight of } C} \longrightarrow \underbrace{W}_{\text{weight of } S} \times \underbrace{F(2 \times 2)}_{F\langle \chi_S, \chi_C \rangle} \times \underbrace{W}_{\text{weight of } C \setminus S}$

Functor:	$G^{(-)}$	$\mathcal{B}_\mathrm{f}$	$\mathcal{D}$	$\mathcal{P}$	F_Σ
Labels L:	G	$\mathbb{N}$	$[0,1]$	1	$\mathbb{N}$
Weights W:	$G^{(2)}$	$\mathcal{B}_\mathrm{f}2$	$\mathcal{D}2$	$\mathbb{N}$	$F_\Sigma 2$
$w(C)$, $C \subseteq Y$:	$G\chi_C$	$\mathcal{B}_\mathrm{f}\chi_C$	$\mathcal{D}\chi_C$	$\lvert C \cap (-) \rvert$	$F_\Sigma \chi_C$

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