An Efficient Coalgebraic Paige Tarjan Algorithm

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Joint work with:
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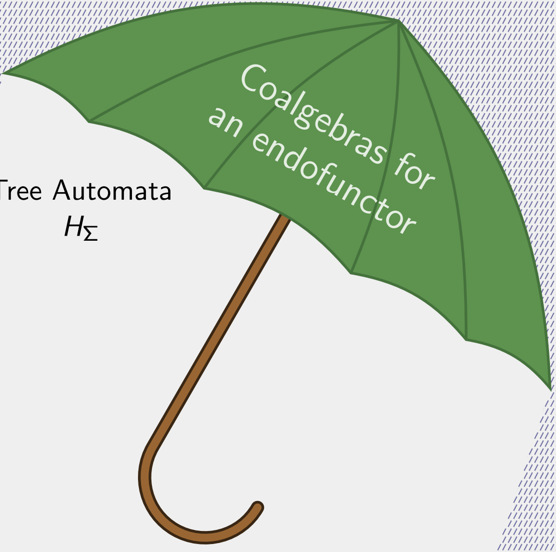
Calco Early Ideas
June 16, 2017

Efficient Minimization?



Tree Automata
H_Σ

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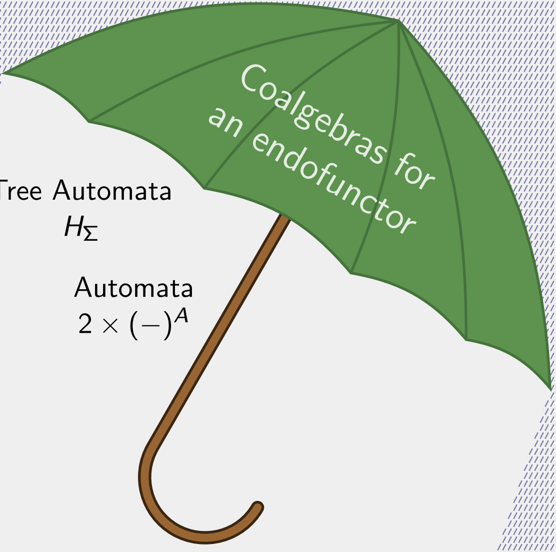

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Automata
$2 \times (-)^A$

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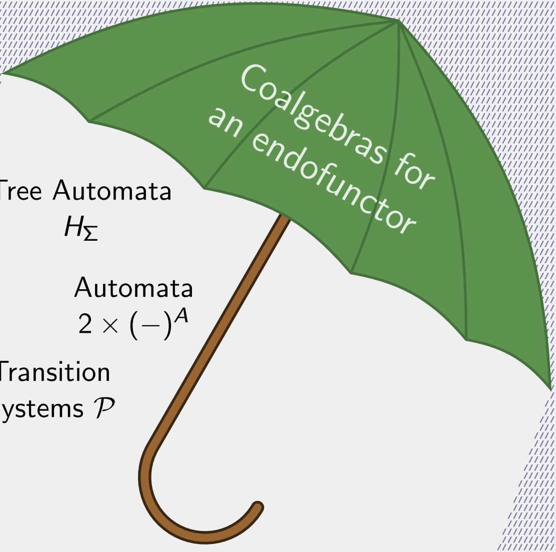

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Systems $\mathcal{P}$

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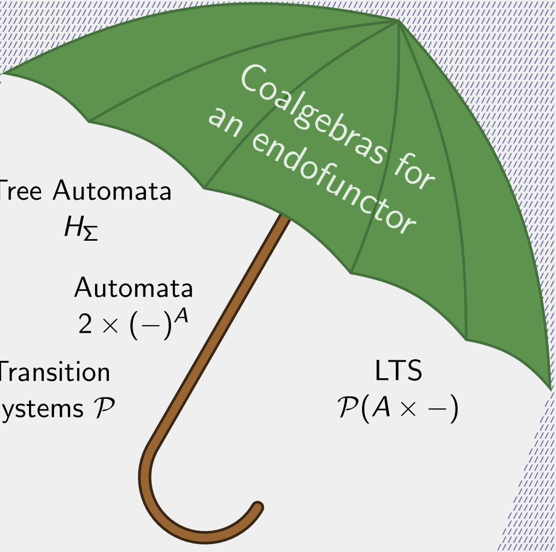

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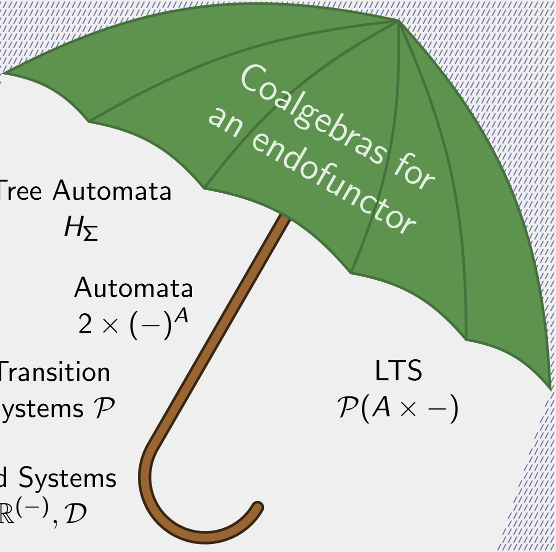

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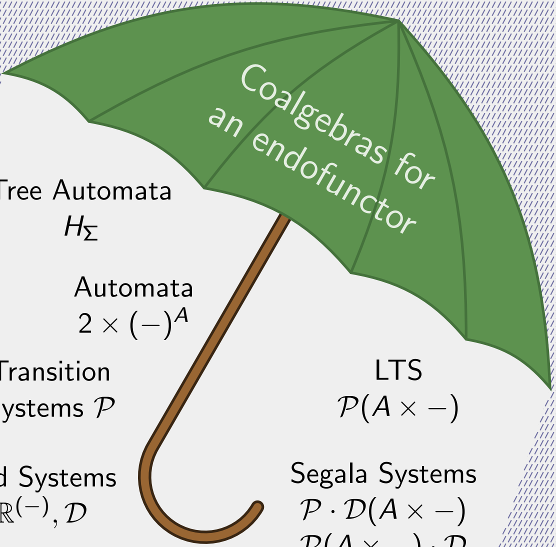

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Segala Systems
$\mathcal{P} \cdot \mathcal{D}(A \times -)$
$\mathcal{P}(A \times -) \cdot \mathcal{D}$

Efficient Minimization?

1. Assume everything equivalent

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4. Pick some of the new information

$$\begin{array}{c} X \\ {\scriptstyle !}\downarrow \\ 1 \end{array} \quad \longrightarrow \quad \begin{array}{c} Q := \ker a \\ \downarrow\downarrow \\ X \\ \downarrow{\scriptstyle a} \\ A \end{array}$$

refine further

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$$
\begin{tikzcd}
 & & P := \ker(\xi \cdot Ha) \arrow[d, shift left] \arrow[d, shift right] \\
 & Q := \ker a \arrow[d, shift left] \arrow[d, shift right] & X \arrow[d, "\xi"] \\
X \arrow[d, "!"'] & X \arrow[d, "a"] & HX \arrow[d, "Ha"] \\
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heuristic

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consider $X \xrightarrow{a'} A \times B$

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Heuristic id on X/P:

Use all immediately

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heuristic

Heuristic id on X/P:

Use all immediately

Heuristic in Set:

Process “smaller half”

Assume

Finitely complete, H mono-preserving, (RegularEpi,Mono)-factorisations

Theorem (Correctness)

$$\begin{tikzcd} X \arrow[r, "\xi"] \arrow[d, two heads] & HX \arrow[d] \\ X/P_i \arrow[r] & H(X/Q_i) \end{tikzcd}$$

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If $P_i \cong Q_i$, then this

1. is a coalgebra
2. has no proper quotient

Incremental partitions

$$
\begin{tikzcd}
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X \arrow[d, "a"] \\
A
\end{tikzcd}
$$

Incremental partitions

$$\begin{tikzcd}
Q := \ker a \arrow[d, shift left] \arrow[d, shift right] & Q \cap \ker b \arrow[d, shift left] \arrow[d, shift right] \\
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A & A \times B
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\end{tikzcd}
$$

Question: When is $\ker H\langle a, b\rangle = \ker\langle Ha, Hb\rangle$?

Incremental partitions

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Theorem: In Set, $\ker H\langle a, b\rangle = \ker\langle Ha, Hb\rangle$ if

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$$
\begin{array}{c} H(L+R) \\ \downarrow \\ H(L+1) \times H(1+R) \end{array} \quad \text{monic} \qquad \text{and}
$$

“zippable”

Incremental partitions

$$
\begin{array}{cccc}
 & & P := \ker(\xi \cdot Ha) & {\color{red}???} \\
 & & \downdownarrows & \downdownarrows \\
Q := \ker a & Q \cap \ker b & X & X \\
\downdownarrows & \downdownarrows & \downarrow{\scriptstyle \xi} & \downarrow{\scriptstyle \xi} \\
X & X & HX & HX \\
\downarrow{\scriptstyle a} & \downarrow{\scriptstyle \langle a,b\rangle} & \downarrow{\scriptstyle Ha} & \downarrow{\scriptstyle H\langle a,b\rangle} \\
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\end{array}
\quad \text{monic} \quad \text{and} \quad
\begin{array}{c}
\ker a \cup \ker b \\
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"zippable"

Setting for complexity analysis

Category:	Heuristic:	Functor:
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- compute "smaller half" heuristic in linear time

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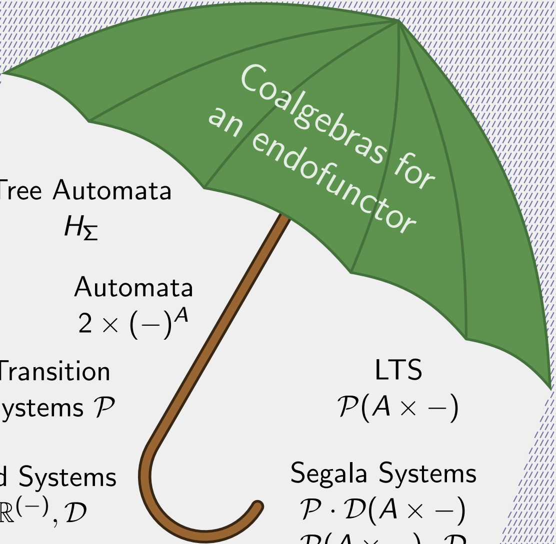
Theorem

Overall complexity: $\mathcal{O}((m+n) \cdot \log n)$ for $n = |X|$, $m = \sum_{x \in X} |\flat\xi(x)|$



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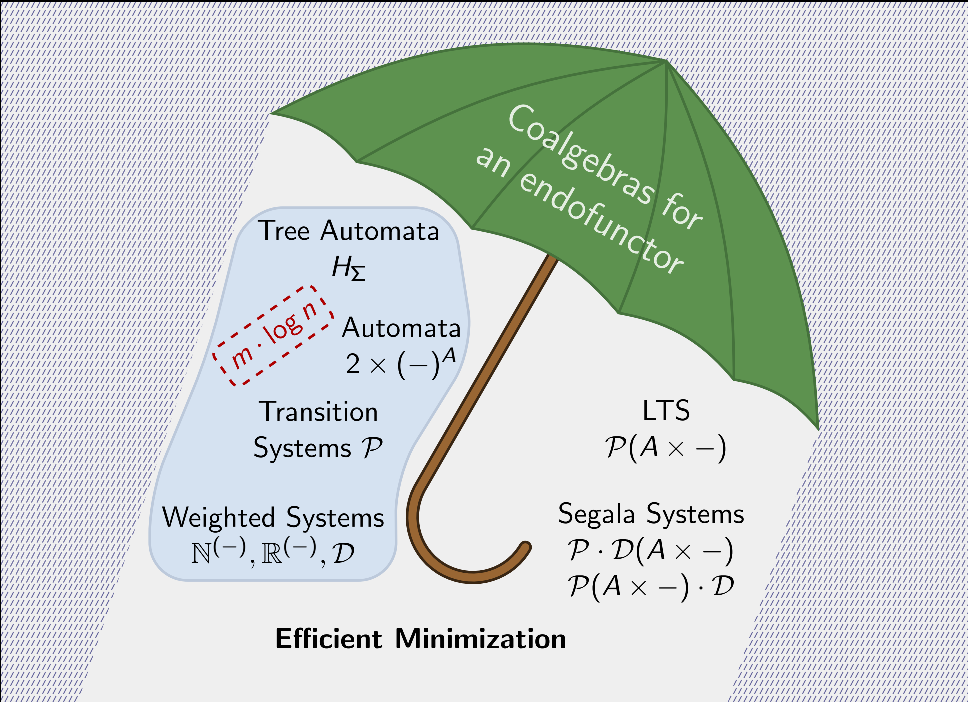
Coalgebras for an endofunctor
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$m \cdot \log n$
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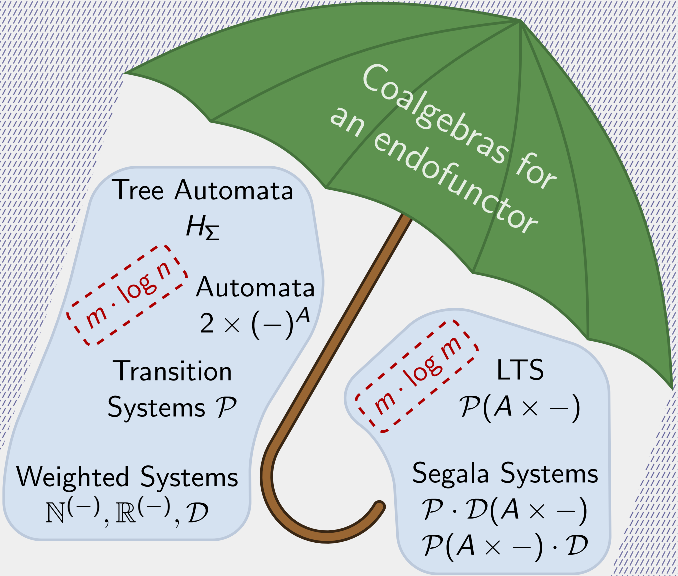

Efficient Minimization

Functors H zippable, if

$$H(L+R) \xrightarrow{\text{unzip}} H(L+1) \times H(1+R) \text{ is monic.}$$

E.g. Id, Constants, $\times$, $+$, $\hookrightarrow$, $M^{(-)}$, part. additive

Examples for sets $L = \{a_1, a_2, a_3\}$, $R = \{b_1, b_2\}$, $1 = \{\_\}$

$$a_1\, a_2\, b_1\, a_3\, b_2 \overset{\text{unzip}}{\longmapsto} (a_1 a_2 \_ a_3 \_,\ \_\,\_\, b_1\, \_\, b_2)$$

$(-)^*$ is zippable

$$\{a_1, a_2, b_1\} \overset{\text{unzip}}{\longmapsto} (\{a_1, a_2, \_\}, \{\_, b_1\})$$

$\mathcal{P}$ is zippable

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$\mathcal{P}$ is zippable

$$\{\{a_1, b_1\}, \{a_2, b_2\}\} \overset{\text{unzip}}{\longmapsto} (\{\{a_1, \_\}, \{a_2, \_\}\},\ \{\{\_, b_1\}, \{\_, b_2\}\}) \overset{\text{unzip}}{\longmapsfrom} \{\{a_1, b_2\}, \{a_2, b_1\}\}$$

$\mathcal{P}\mathcal{P}$ is not zippable

~~Composition~~

~~Quotients~~

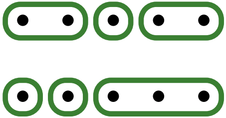
$$A \xleftarrow{a} X \xrightarrow{b} B$$

$\ker a \cup \ker b$ a kernel in Set

$\Leftrightarrow$ $\ker a \cup \ker b$ transitive

$\Leftrightarrow$ $\forall x \in X : [x]_a \subseteq [x]_b$ or $[x]_a \supseteq [x]_b$

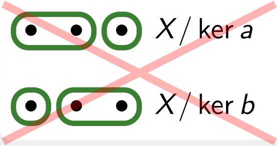
Example



$X/\ker a$

$X/\ker b$

Non-Example



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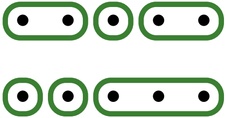

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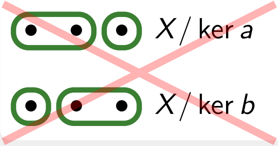

$X / \ker a$

$X / \ker b$

Process smaller half for $X \xrightarrow{f} F \xrightarrow{g} G$

Find $x \in X$, with $S := [x]_f$, $C := [x]_{gf}$, such that $2 \cdot |S| \leq |C|$.
Return $\langle \chi_S, \chi_C \rangle : X \to 2 \times 2$

Functor encoding

- internal weights W, $w : HX \to \mathcal{P}X \to W$
- edge labels L
- $\flat : HX \to \mathcal{B}_{\mathrm{f}}(L \times X)$
- update : $\mathcal{B}_{\mathrm{f}}(L) \times W \longrightarrow W \times H(2 \times 2) \times W$

edges into S weight of C weight of S $H\langle \chi_S, \chi_C \rangle$ weight of $C \setminus S$

Functor:	$G^{(-)}$	$\mathcal{B}_{\mathrm{f}}$	$\mathcal{D}$	$\mathcal{P}$	H_Σ
Labels L:	G	$\mathbb{N}$	$[0,1]$	1	$\mathbb{N}$
Weights W:	$G^{(2)}$	$\mathcal{B}_{\mathrm{f}}2$	$\mathcal{D}2$	$\mathbb{N}$	$H_\Sigma 2$
$w(C)$, $C \subseteq Y$:	$G\chi_C$	$\mathcal{B}_{\mathrm{f}}\chi_C$	$\mathcal{D}\chi_C$	$\lvert C \cap (-) \rvert$	$H_\Sigma \chi_C$

Future work

- Implementation & Benchmarking
- $\mathcal{O}(m \cdot \log n)$ on $\mathcal{P}(A \times -)$
- $\ker\langle Ha, Hb\rangle = \ker H\langle a, b\rangle$ outside of Set?
- Further functors, e.g. monotone neighbourhoods.