MBProg Summary

Florian Guthmann
florian.guthmann@fau.de

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Semantics for Computation

Evaluation Strategies

α -conversion $C[\lambda x.t] \to_\alpha C[\lambda y.t[y/x]]$

β -reduction $C[(\lambda x.t)s] \to_\beta C(t[s/x])$

η -reduction $C[\lambda x.fx \to_\eta C[f]]$

Confluence (Church-Rosser) Independent reductions starting from the same term can always be joined.

Church Numerals

$0 = \lambda f.\lambda z.z$

$1 = \lambda f.\lambda z.fz$

$2 = \lambda f.\lambda z.ffz$

Standard

We impose a *left-most-outermost* evaluation order.

Standardization Theorem If $s \to^*_{\alpha\beta} t$ and t is $\alpha\beta$ -normal, then $s \to^*_{\text{so}} t$ and t is so-normal.

Call-by-Name (lazy)

small-step

- no more rewriting under λ
- all terms are closed

$$\frac{}{(\lambda x.p)q \to p[q/x]} \qquad \frac{p \to p'}{pq \to p'q}$$

big-step

$$\frac{}{\lambda x.p \Downarrow \lambda x.p} \qquad \frac{q \Downarrow \lambda x.p' \quad p'[q/x] \Downarrow v}{pq \Downarrow v}$$

Call-by-Value (eager)

Value A value is a term of the form $\lambda x.t$

small-step

$$\frac{p \to p'}{pq \to p'q} \qquad \frac{q \to q' \quad p \text{ is a value}}{pq \to pq'} \qquad \frac{q \text{ is a value}}{(\lambda x.p)q \to p[q/x]}$$

big-step

$$\frac{}{\lambda x.p \Downarrow \lambda x.p} \qquad \frac{p \Downarrow \lambda x.p' \quad q \Downarrow q' \quad p'[q'/x] \Downarrow c}{pq \Downarrow c}$$

PCF

Simply-Typed Lambda-Calculus

$$\text{Type} := \underbrace{A, B, C, \ldots}_{\text{base types}} | \underbrace{1}_{\text{unit type}} | A \times B | A \to B$$

$\Omega = (\lambda x.xx)(\lambda x.xx)$ is not typable. Thus, $\to_{\alpha\beta}$ is strongly normalising for simply-typed lambda-calculus.

We obtain *PCF* by:

- adding the fixpoint combinator $Y_A : (A \to A) \to A$ for every type A
- fixing Nat and Bool as the base types
- postulating the arithmetic and logical operations

Contextual Equivalence A term context C is of *ground type* if its type is either Nat, Bool or 1. Two PCF terms $\Gamma \vdash s : A$ and $\Gamma \vdash t : A$ are *contextually equivalent*, if for all contexts C of ground type and for all values v, $C[s] \Downarrow v$ iff $C[t] \Downarrow v$

Operational Semantics

Call-by-Name A value is

- a term of base or unit type
- a pair of closed terms
- a closed term of the form $\lambda x.t$

Call-by-Value

- a term of base or unit type
- a pair of values
- a closed term of the form $\lambda x.t$

Denotational Semantics

Partial Order A partial order $(A,\sqsubseteq)$ is a relation, such that:

- $a \sqsubseteq a$ (reflexivity)
- $a \sqsubseteq b \wedge b \sqsubseteq c \Rightarrow a \sqsubseteq c$ (transitivity)
- $a \sqsubseteq b \wedge b \sqsubseteq a \Rightarrow a = b$ (antisymmetry)

Complete Partial Order A complete partial order (pre-domain) is a partial order $(A,\sqsubseteq)$ such that for any infinite chain

$$a_1 \sqsubseteq a_2 \sqsubseteq ...$$

there is a least upper bound a such that

- $\forall i. a_i \sqsubseteq a$
- $\forall i. a_i \sqsubseteq b \Rightarrow a \sqsubseteq b$

We denote such a by $\bigsqcup_i a_i$

Domain A pointed cpo (*domain*) is a cpo that contains an element $\bot$ such that $\forall a \in A. \bot \sqsubseteq a$

Monotonicity A function $f: A \to B$ between partial orders is *monotone* if

$$a \sqsubseteq b \Rightarrow f(a) \sqsubseteq f(b)$$

Scott-continuity A monotone function $f: A \to B$ between cpos is continuous if for any chain $a_1 \sqsubseteq a_2 \sqsubseteq ...$

$$f\left(\bigsqcup_i a_i\right) = \bigsqcup_i f(a_i)$$

Strictness A function $f: a \to B$ is *strict* if $f(\bot) = \bot$

Products of Predomains

$$A \times B = \{(a,b) | a \in A, b \in B\} \qquad (a_1,b_1) \sqsubseteq (a_2,b_2) \text{if} a_1 \sqsubseteq a_2 \wedge b_1 \sqsubseteq b_2$$

Pairing is continuous: $\bigsqcup_i (a_i,b_i) = \left(\bigsqcup_i a_i, \bigsqcup_j b_j\right)$

Lifting predomains and functions

$$A_\bot = A \uplus \bot = (\star, a) | a \in A \cup (\bot, \star) \qquad a \sqsubseteq b \text{ if } a = \bot \text{ or } a \in A, b \in A \text{ and } a \sqsubseteq b$$

$$f^*(x) = \begin{cases} f(y) & \text{if } x = \lfloor y \rfloor \\ \bot & \text{if } x = \bot \end{cases}$$

Function Spaces Let $(A,\sqsubseteq)$, $(B,\sqsubseteq)$ be predomains. $(A \to B, \sqsubseteq)$ is the function space predomain where:

$$A \to B = \{f: A \to B \,|\, f \text{ is continuous}\} \qquad f \sqsubseteq g \Leftrightarrow \forall x. f(x) \sqsubseteq g(x)$$

We define:

$$\begin{aligned} \text{curry} &: (A \times B \to C) \to (A \to (B \to C)) \\ \text{uncurry} &: (A \to (B \to C)) \to (A \times B \to C) \\ \text{ev} &: (A \to B) \times A \to B \end{aligned}$$

If B is a domain, so is $A \to B$ with the bottom element being $\lambda x. \bot$

Kleene's Fixpoint Theorem Let f be a continuous function $f: D \to D$ over a domain D

- least fixpoint: $\exists \mu f \in D$ such that $f(\mu f) = \mu f$ and $\forall x \in D. f(x) = x \Rightarrow \mu f \sqsubseteq x$
- $\mu f = \bigsqcup_i f^i(\bot)$, where $f^0(x) = \bot, f^{i+1}(x) = f(f^i(x))$
- least prefixpoint: $f(\mu f) \sqsubseteq \mu f$ and $\forall x \in D. f(x) \sqsubseteq x \Rightarrow \mu f \sqsubseteq x$

CBN

Soundness A denotational semantics is *sound* if

$$p \Downarrow v \Rightarrow [\![p]\!] = v$$

Adequacy A denotational semantics is *adequate* if for p of ground type

$$[\![p]\!] = v \Rightarrow p \Downarrow v$$

for every *value* v

Compositionality

$$[\![C[t]]\!] = [\![C]\!][[\![t]\!]]$$

$$
\begin{aligned}
[\![1]\!] &= 1_\perp \\
[\![\mathrm{Nat}]\!] &= \mathrm{Nat}_\perp \\
[\![\mathrm{Bool}]\!] &= \mathrm{Bool}_\perp \\
[\![A \times B]\!] &= [\![A]\!] \times [\![B]\!] \\
[\![A \to B]\!] &= [\![A]\!] \to [\![B]\!]
\end{aligned}
$$

Given a term in context $\Gamma \vdash t : A$ where $\Gamma = x_1 : A_1, \ldots, x_n : A_n$ the semantics $[\![\Gamma \vdash t : A]\!]$ is a continuous function $[\![A_1]\!] \times \cdots \times [\![A_n]\!] \to [\![A]\!]$ ($[\![\ldots]\!]_\rho = [\![\ldots]\!](\rho)$)

$$
\begin{aligned}
[\![\Gamma \vdash x_i : A_i]\!]_\rho &= \mathrm{proj}_i(\rho) \\
[\![\Gamma \vdash \star : 1]\!]_\rho &= \lfloor \star \rfloor \\
&\vdots \\
[\![\Gamma \lambda x.t : A \to B]\!]_\rho &= (\mathrm{curry}[\![\Gamma, x : A \vdash t : B]\!])(\rho) \\
[\![\Gamma st : B]\!]_\rho &= \mathrm{ev}([\![\Gamma \vdash s : A \to B]\!]_\rho,]\!]\Gamma \vdash t : A]\!]_\rho \\
[\![\Gamma \vdash Y_A]\!] &= \mu
\end{aligned}
$$

CBV

Full Abstraction

The implication $p =_{ctx} q \Rightarrow [\![p]\!] = [\![q]\!]$ is called full abstraction. It would mean that operational semantics and denotational semantics agree as far as program equivalence. However consider the `por` function. It is not definable in PCF, but it is a continuous function and thus can be used in the denotational semantics.

$$
\begin{aligned}
\texttt{por}(\mathrm{True}, x) &= \mathrm{True} \\
\texttt{por}(x, \mathrm{True}) &= \mathrm{True} \\
\texttt{por}(\mathrm{False}, \mathrm{False}) &= \mathrm{False} \\
\texttt{por}(x, y) &= \perp
\end{aligned}
$$

We can construct a function $t : \mathrm{Bool} \to (\mathrm{Bool} \to \mathrm{Bool} \to \mathrm{Bool}) \to \mathrm{Bool}$ in PCF that tests if a given function is `por`.

Category Theory

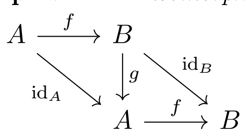
Category A Category $\mathcal{C}$ consist of a collection of objects $\mathrm{Ob}(\mathcal{C})$ and a collection of morphisms $\mathrm{Hom}_\mathcal{C}(A,B)$ for any $A, B \in \mathrm{Ob}(\mathcal{C})$ such that:

- for every $A \in \mathrm{Ob}(\mathcal{C})$ there is an *identity morphism* $id_A \in \mathrm{Hom}_\mathcal{C}(A,A)$
- for any $f \in \mathrm{Hom}_\mathcal{C}(B,C)$ and $g \in \mathrm{Hom}_\mathcal{C}(A,B)$ we can form a *composition* $f \circ g \in \mathrm{Hom}_\mathcal{C}(A,C)$
- $\mathrm{id} \circ f = f$ (left identity)
- $f \circ \mathrm{id} = f$ (right identity)
- $(f \circ g) \circ h = f \circ (g \circ h)$ (associativity)

Terminal Object A terminal object is an object $1 \in \mathrm{Ob}(\mathcal{C})$ such that for any $A \in \mathrm{Ob}(\mathcal{C})$, there is a unique morphism $!_A : A \to 1$.

Initial Object An initial object is an object $0 \in \mathrm{Ob}(\mathcal{C})$ such that for any $A \in \mathrm{Ob}(\mathcal{C})$, there is a unique morphism $¡_A : 0 \to A$.

Isomorphism An *isomorphism* between objects A and B in a category $\mathcal{C}$ is a pair of morphisms $f : A \to B, g : B \to A$ such that:

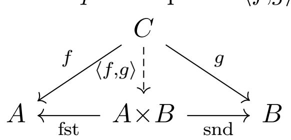


Cartesian Category A *cartesian category* is a category with a terminal object an binary products

Products and Coproducts

Binary Products

A product of objects A, B in a category $\mathcal{C}$ is a triple $(A \times B, \mathrm{fst}, \mathrm{snd})$ such that for any $C \in \mathrm{Ob}(\mathcal{C})$ and $f : C \to A$ and $g : C \to B$ there is a *unique* morphism $\langle f, g \rangle : C \to A \times B$:



Coproducts

A coproduct of objects A,B in a category $\mathcal{C}$ is a triple $(A+B,\text{inl},\text{inr})$ such that for any $f:A\to C$ and $g:B\to C$ there is a *unique* morphism $[f,g]:A+B\to C$:

$$\begin{array}{ccccc} & & C & & \\ & \overset{f}{\nearrow} & \uparrow{\scriptstyle [f,g]} & \overset{g}{\nwarrow} & \\ A & \xrightarrow[\text{inl}]{} & A+B & \xleftarrow[\text{inr}]{} & B \end{array}$$

Functor

A covariant Functor F between categories $\mathcal{C}$ and $\mathcal{D}$ is a correspondence sending any $A\in\text{Ob}(\mathcal{C})$ to $FA\in\text{Ob}(\mathcal{D})$ and any $f\in\text{Hom}_{\mathcal{C}}(A,B)$ to $Ff\in\text{Hom}_{\mathcal{D}}(FA,FB)$ such that:

$F(\text{id}_A)=\text{id}_{FA}$ $\qquad$ $F(f\circ g)=F(f)\circ F(g)$

Example: Forgetful Functor

$G:\text{Cpo}\to\text{Set}$ $\qquad$ $G(A,\sqsubseteq)=A$ $\qquad$ $G(f)=f$

Example: Endofunctor An *endofunctor* is a functor from a category into itself.

Contravariant Functor A functor $F:\mathcal{C}^{\text{op}}\to D$ is a contravariant functor from $\mathcal{C}$ to $\mathcal{D}$.

Bifunctor A bifunctor is a functor $\mathcal{C}\times\mathcal{D}\to\mathcal{E}$.

Natural Transformations

Let $\mathcal{C}$,$\mathcal{D}$ be categories and F,$G:\mathcal{C}\to\mathcal{D}$ be functors. A *natural transformation* $\vartheta:F\to G$ is a family of morphisms

$(\vartheta_C:FC\to GC)_{c\in\text{Ob}(\mathcal{C})}$

such that $\forall f:C\to C'$ in $\mathcal{C}$:

$$\begin{array}{ccc} FC & \xrightarrow{Ff} & FC' \\ \downarrow{\scriptstyle \vartheta_C} & & \downarrow{\scriptstyle \vartheta_{C'}} \\ GC & \xrightarrow{Gf} & GC' \end{array}$$

Monad

Kleisli Triple

A *Monad* in a category $\mathcal{C}$ is given by a triple $(T,\eta,-^*)$ where:

- $T:\text{Ob}(\mathcal{C})\to\text{Ob}(\mathcal{C})$
- unit: η is a family $(\eta_X:X\to TX)_{x\in\text{Ob}(\mathcal{C})}$
- Kleisli lifting: for any $f:A\to TB$, $f^*:TA\to TB$

such that:

$\eta^*=\text{id}$ $\qquad$ $f^*\eta=f$ $\qquad$ $(f^*g)^*=f^*g^*$

Kleisli Category

From Endofunctor and Natural Transformation

A *Monad* in a category $\mathcal{C}$ consists of an endofunctor $T:\mathcal{C}\to\mathcal{C}$ an natural transformations

- unit: $\eta:\text{Id}\to T$
- multiplication: $\mu:TT\to T$

such that:

$\mu\circ\mu_T=\mu\circ T\mu$ $\qquad$ $\mu\circ\eta_T=\text{id}=\mu\circ T\eta$

$$\begin{array}{ccc} TTTX & \xrightarrow{\mu_{TX}} & TTX \\ \downarrow{\scriptstyle T\mu_X} & & \downarrow{\scriptstyle \mu_X} \\ TTX & \xrightarrow{\mu_X} & TX \end{array} \qquad \begin{array}{ccccc} TX & \xrightarrow{\eta_{TX}} & TTX & \xleftarrow{T\eta_X} & TX \\ & \underset{\text{id}_{TX}}{\searrow} & \downarrow{\scriptstyle \mu_X} & \underset{\text{id}_{TX}}{\swarrow} & \\ & & TX & & \end{array}$$

Tensorial Strength

Cartesian Closure A category $\mathcal{C}$ is *cartesian closed*(CCC) if it is cartesian, and for any objects B and C there is an object B^C, called an exponential, for which

$\text{curry}:\text{Hom}(A\times B,C)\cong\text{Hom}(A,C^B)$

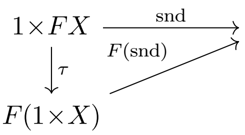
which is natural in A such that

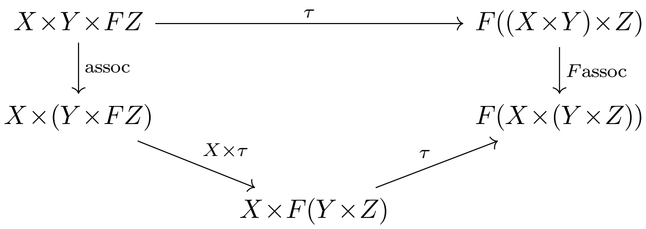
$$\begin{array}{ccc} \text{Hom}(A\times B,C) & \xrightarrow{\text{curry}} & \text{Hom}(A,C^B) \\ \downarrow{\scriptstyle \text{Hom}(f\times B,C)} & & \downarrow{\scriptstyle \text{Hom}(f,C^B)} \\ \text{Hom}(A'\times B,C) & \xrightarrow{\text{curry}} & \text{Hom}(A',C^B) \end{array}$$

We can generalize the CBV semantics of PCF by:

- replace $(-)_\perp$ with T
- replace "let" with "do"
- replace $\lfloor - \rfloor$ with `return`

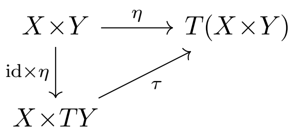
Strong Functor A strong Functor is a functor $F:\mathcal{C}\to\mathcal{D}$ between cartesian categories $\mathcal{C}$ and $\mathcal{D}$, plus *strength*, which is a natural transformation $\tau_{A,B}:A\times FB\to F(A\times B)$ such that

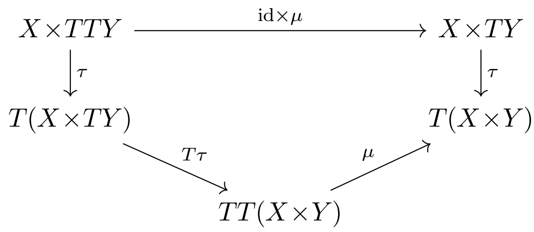




Strong Monads

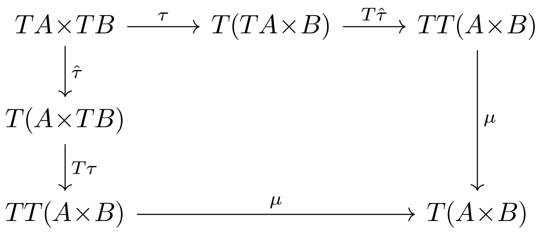
A monad is *strong* if it is strong as a functor and η,μ are strong natural transformations.





Commutative Monads

A strong monad T is commutative if



In other words,

$\mathrm{do}\, x=p;\mathrm{do}\, y=q;\mathrm{return}\langle x,y\rangle == \mathrm{do}\, y=q;\mathrm{do}\, x=p;\mathrm{return}\langle x,y\rangle$

Monoidal Categories

A category $\mathcal{C}$ is *monoidal* if there exists:

- a bifunctor $C\times C\to C$ (tensor product)
- an object I (unit object)
- natural transformations

$\alpha_{A,B,C}:A\otimes(B\otimes C)\cong(A\otimes B)\otimes C \qquad \lambda_A:I\otimes A\cong A \qquad \rho_A:A\otimes I\cong A$

A *monoid* in a monoidal Category $\mathcal{C}$ is a triple $(M,\varepsilon,\odot)$ where M is an object in $\mathcal{C}$, $\odot$ is a morphism $M\otimes M\to M$ and ε is a morphism $I\to M$ such that: