Oriented Trees

TAoCP-Seminar

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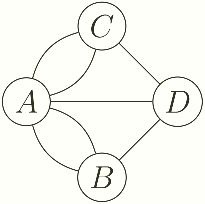
Graphs

A *graph* $\mathcal{G} = \langle V, E \rangle$ consists of

- A set of *vertices* (points) V
- A set of *edges* E

We call vertices G and H *adjacent*, if there is an edge between them.



$$V = \{A, B, C, D\}$$

$$E = \{e_0^{AB}, e_1^{AB}, e_0^{AC}, e_1^{AC}, e^{AD}, e^{BD}, e^{CD}\}$$

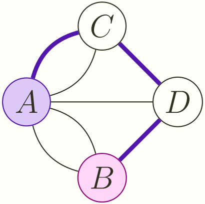
Paths

Let V_0 and V_n be vertices and $n \geq 0$. Then

$$(V_0, V_1, \ldots, V_n)$$

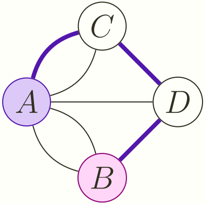
is a *path* of length n if

V_k is adjacent to V_{k+1}, for $0 \leq k < n$

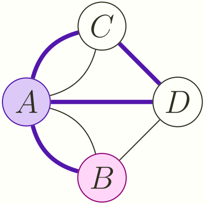


(A, C, D, B)

A path is *simple*, if $V_0, V_1, \ldots, V_n$ are distinct.

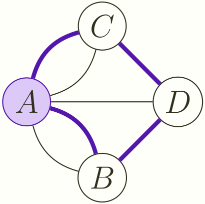


$P = (A, C, D, B)$
simple



$P = (A, C, D, A, B)$
not simple

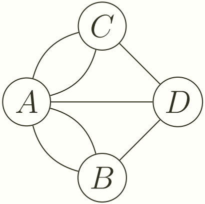
A *cycle* is a path from a vertex to itself (i.e. where $V_0 = V_n$) and where $V_1, \ldots, V_{n-1}$ are distinct.



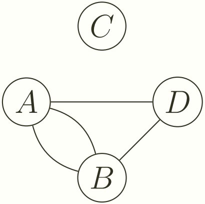
$$P = (A, B, D, C, A)$$

Connectedness

A graph is *connected*, if there is a path between any two vertices.



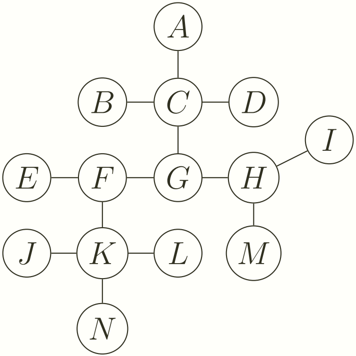
connected



not connected

Free Tree

A *free tree* is a connected graph with no cycles.



Adapted from Fig.30

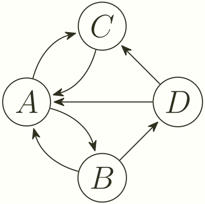
Directed Graphs

Definition

A directed graph $\mathcal{G} = \langle V, A \rangle$ consist of

- A set of *vertices* V
- A set of *arcs* A

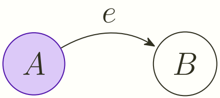
We will commonly abbreviate directed graph as *digraph*.



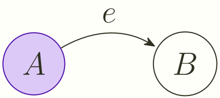
$$V = \{A, B, C, D\}$$
$$A = \{e^{AB}, e^{BA}, e^{AC}, e^{CA}, e^{DA}, e^{BD}, e^{DC}\}$$

Each arc e has an *initial* vertex:

$\mathrm{init}(e) = A$

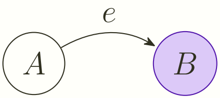


Each arc e has an *initial* vertex:



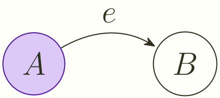
$$\mathrm{init}(e) = A$$

And a *final* vertex:



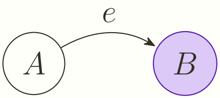
$$\mathrm{fin}(e) = B$$

Each arc e has an *initial* vertex:

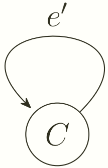


$\mathrm{init}(e) = A$

And a *final* vertex:



$\mathrm{fin}(e) = B$

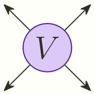


$\mathrm{init}(e') = \mathrm{fin}(e') = C$

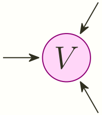
For a vertex V, we define

out-degree
: the number of arcs e such that $\mathrm{init}(e) = V$

in-degree
: the number of arcs e such that $\mathrm{fin}(e) = V$



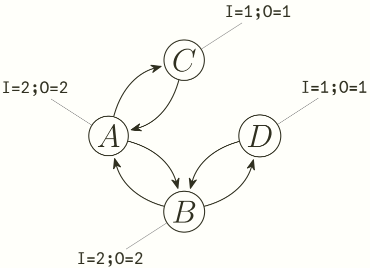
$\text{out-degree}(V) = 4$



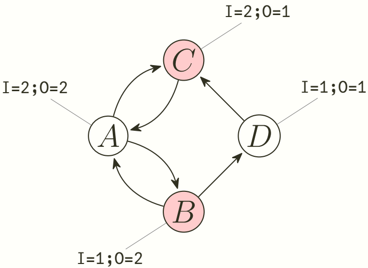
$\text{in-degree}(V) = 3$

Balance

A directed graph $\mathcal{G} = \langle V, A \rangle$ is *balanced* if for every $v \in V$, $\text{in-degree}(v) = \text{out-degree}(v)$



balanced

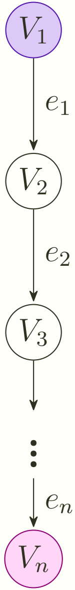


not balanced

Oriented path

Let $e_1, e_2, \ldots, e_n$, $n \geq 1$ be arcs in a digraph. Then $(e_1, e_2, \ldots, e_n)$ is an *oriented path* of length n from V to V' if

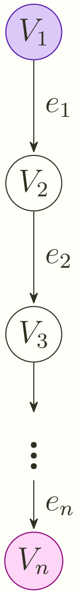
- $\text{init}(e_1) = V$
- $\text{fin}(e_n) = V'$
- $\text{fin}(e_i) = \text{init}(e_{i+1})$ for $1 \leq i < n$



Oriented path

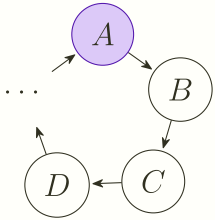
An oriented path is *simple* if

- $\text{init}(e_1), \ldots, \text{init}(e_n)$ are distinct
- $\text{fin}(e_1), \ldots, \text{fin}(e_n)$ are distinct

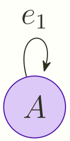


Oriented cycle

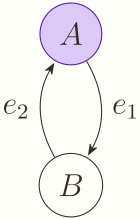
An *oriented cycle* is a simple oriented path from a vertex to itself.



$$\text{init}(e_1) = A = \text{fin}(e_n)$$



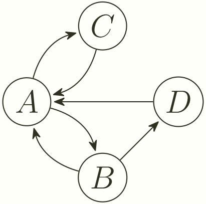
(e_1) is an oriented path of length 1



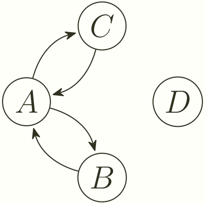
(e_1, e_2) is an oriented path of length 2

Connectedness

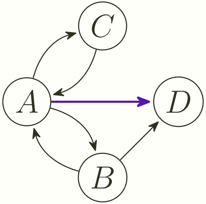
A digraph is *strongly connected* if there is an oriented path from G to H for any two vertices $G, H, \quad G \neq H$



connected



not connected

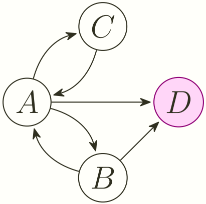


not connected

(weakly connected)

Root

A *root* in a digraph $\mathcal{G}$ is a vertex R such that there is an oriented path from V to R for any vertex V in $\mathcal{G}$, $V \neq R$

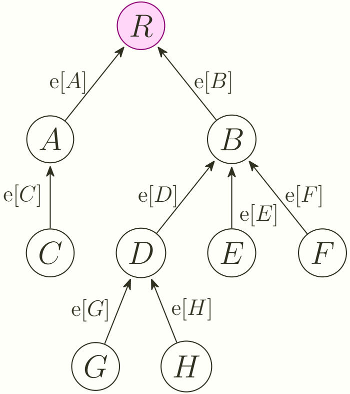


$$R = D$$

Oriented Tree

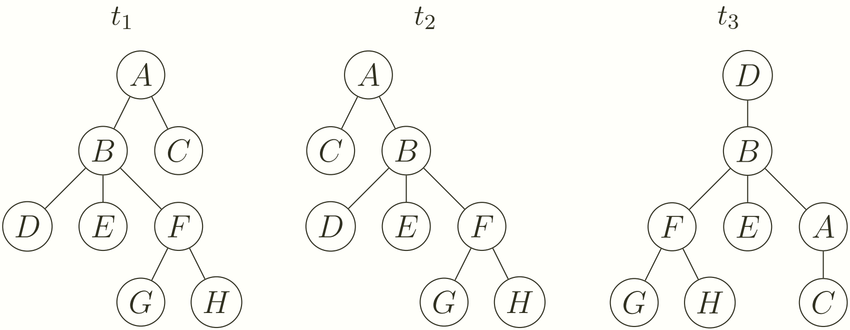
An *oriented tree* is a digraph with a vertex R such that

- each vertex $V \neq R$ is the initial vertex of exactly one arc $\mathrm{e}[V]$
- R is the inital vertex of no arc
- R is a root



Adapted from Fig. 34

Types of Trees



As oriented trees[1] $t_1 = t_2 \neq t_3$

As free trees: $t_1 = t_2 = t_3$

[1] With arcs pointing “up”

Eulerian Circuit

Definition

An *eulerian circuit* in a digraph $\mathcal{G}$ is an oriented path $(e_1, e_2, \ldots, e_n)$ such that

- each arc in $\mathcal{G}$ occurs exactly once
- $\operatorname{fin}(e_n) = \operatorname{init}(e_1)$

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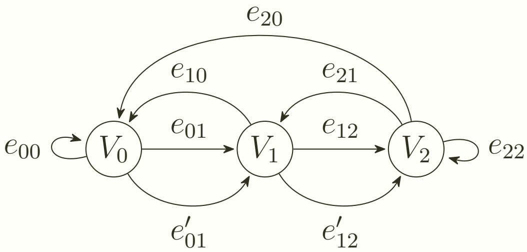
Theorem G

Theorem

A finite, directed graph with no isolated vertices possesses an eulerian circuit iff it is connected and balanced.

Theorem G (Proof)

Fix a balanced and connected Graph $\mathcal{G}$.



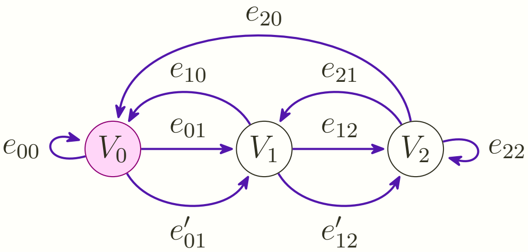
G

$$\text{in-degree}(V_0) = \text{out-degree}(V_0) \quad = 3$$
$$\text{in-degree}(V_1) = \text{out-degree}(V_1) \quad = 3$$
$$\text{in-degree}(V_2) = \text{out-degree}(V_2) \quad = 3$$

Adapted from Fig. 36

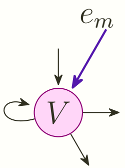
Theorem G (Proof)

Let $P = (e_1, \ldots, e_m)$ be an oriented path of longest possible length that uses no arc twice.
e.g., $P = (e_{01}, e_{12}, e_{22}, e_{20}, e_{00}, e'_{01}, e'_{12}, e_{21}, e_{10})$



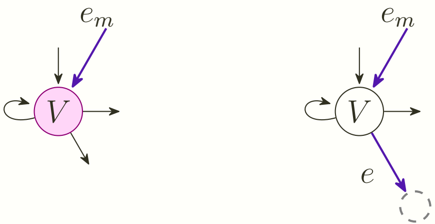
Theorem G (Proof)

$$V = \text{fin}(e_m)$$



Theorem G (Proof)

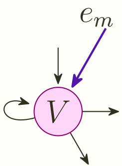
$V = \text{fin}(e_m)$ $\quad\quad$ $P = (e_1, \ldots, e_m, e)$ ↯



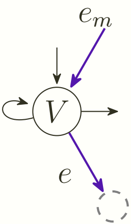
$$e \in \{e_i\}_{i=1,\ldots,m}$$

Theorem G (Proof)

$$V = \mathrm{fin}(e_m)$$

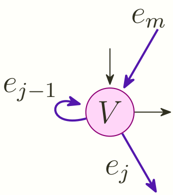


$$P = (e_1, \ldots, e_m, e)\lightning$$



$$e \in \{e_i\}_{i=1,\ldots,m}$$

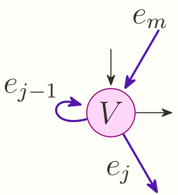
$$\mathrm{init}(e_j) = V$$



$$\mathrm{fin}(e_{j-1}) = V$$

Theorem G (Proof)

G is *balanced*

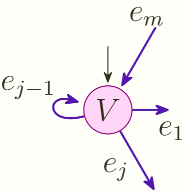


$\text{init} = V$	$\text{fin} = V$
	e_m
e_j	$e_j - 1$
⋮	⋮

$$\text{init}(e_j) = V$$
$$\text{fin}(e_{j-1}) = V$$

Theorem G (Proof)

G is *balanced*



init $= V$	fin $= V$
e_1	e_m
e_j	$e_j - 1$
$\vdots$	$\vdots$

$$\text{init}(e_1) = V = \text{fin}(e_m)$$

$$\text{init}(e_j) = V$$
$$\text{fin}(e_{j-1}) = V$$

$\implies$ P is a *cycle*

Theorem G (Proof)

Now if there exists an arc e not in P

$$\begin{aligned} \operatorname{init}(e) &\neq \operatorname{init}(e_i) \qquad &, i = 1, \ldots, m \\ \operatorname{fin}(e) &\neq \operatorname{fin}(e_i) \qquad &, i = 1, \ldots, m \end{aligned}$$

Theorem G (Proof)

Now if there exists an arc e not in P

$$\operatorname{init}(e) \neq \operatorname{init}(e_i) \qquad , i = 1, \ldots, m$$
$$\operatorname{fin}(e) \neq \operatorname{fin}(e_i) \qquad , i = 1, \ldots, m$$

$\mathcal{G}$ is not connected. $\lightning$
$\implies$ No such arc exists!

Theorem G (Proof)

Now if there exists an arc e not in P

$$\begin{aligned} \text{init}(e) &\neq \text{init}(e_i) \qquad &, i = 1, \dots, m \\ \text{fin}(e) &\neq \text{fin}(e_i) \qquad &, i = 1, \dots, m \end{aligned}$$

$\mathcal{G}$ is not connected. $\lightning$
$\implies$ No such arc exists!

$\implies$ P is an eulerian circuit. $\blacksquare$

Theorem D

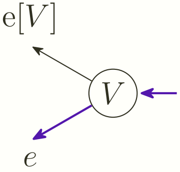
Theorem

Let $\mathcal{G}$ be a balanced, directed graph.
Let $\mathcal{G}'$ be an oriented tree of all vertices and some arcs of $\mathcal{G}$ with root R.
Let e_1 be an arc of $\mathcal{G}$ with $\mathrm{init}(e_1) = R$
Then $P = (e_1, e_2, \ldots, e_m)$ is an eulerian circuit if

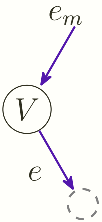
1. *no arc is used twice*
2. *if $e_j = \mathrm{e}[V]$ and there is an arc e with $\mathrm{init}(e) = V$ then $e = e_k, k \leq j$*
3. *if $\mathrm{init}(e) = \mathrm{fin}(e_m)$ then $e = e_k$ for some k*

Theorem D

2. if $e_j = \mathrm{e}[V]$ and there is an arc e with $\mathrm{init}(e) = V$ then $e = e_k, k \leq j$

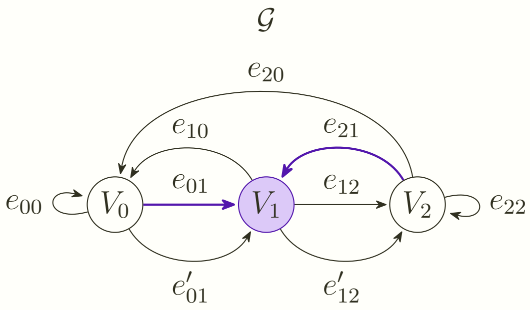


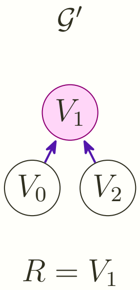
3. if $\mathrm{init}(e) = \mathrm{fin}(e_m)$ then $e = e_k$ for some k



Analogously to *Theorem G*.

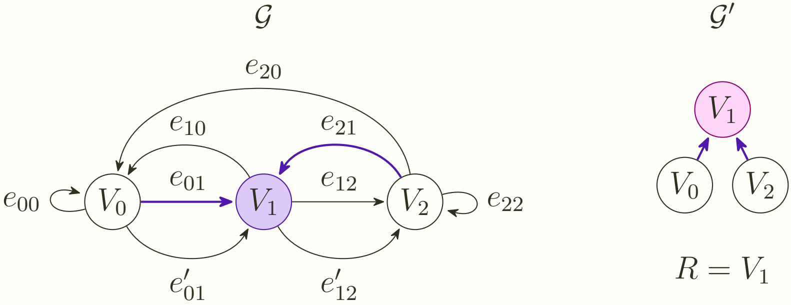
Find an eulerian circuit from e_{12}





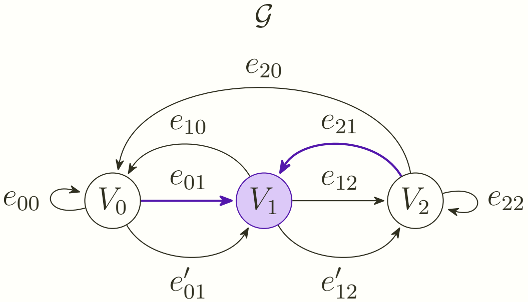
1. $P = (e_{12}, \dots)$

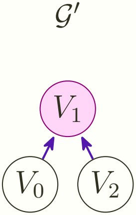
Find an eulerian circuit from e_{12}



1. $P = (e_{12}, \dots)$
2. Choose e_{22}. $P = (e_{12}, e_{22}, \dots)$

Find an eulerian circuit from e_{12}



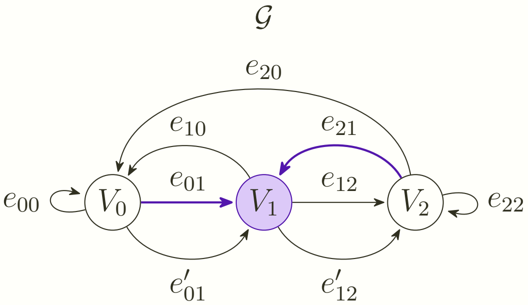


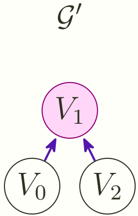
$R = V_1$

1. $P = (e_{12}, \dots)$
2. Choose e_{22}. $P = (e_{12}, e_{22}, \dots)$
3. Choose e_{20}. $P = (e_{12}, e_{22}, e_{20}, \dots)$

⋮

Find an eulerian circuit from e_{12}





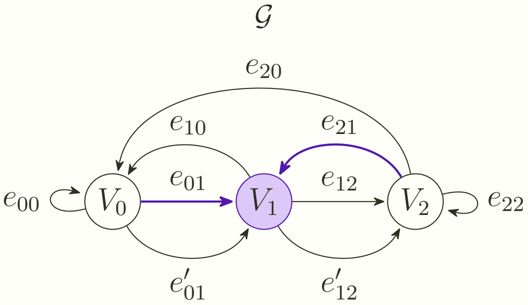
$R = V_1$

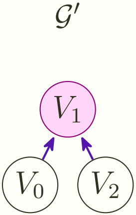
1. $P = (e_{12}, \dots)$
2. Choose e_{22}. $P = (e_{12}, e_{22}, \dots)$
3. Choose e_{20}. $P = (e_{12}, e_{22}, e_{20}, \dots)$

⋮

4. $P = (e_{12}, e_{22}, e_{20}, e_{00}, e'_{01}, e'_{12}, \dots)$
   Now we can choose $e_{21} = \mathrm{e}[V_2]$

Find an eulerian circuit from e_{12}





$R = V_1$

4. $P = (e_{12}, e_{22}, e_{20}, e_{00}, e'_{01}, e'_{12}, \dots)$
   Now we can choose $e_{21} = \mathrm{e}[V_2]$

$P = (e_{12}, e_{22}, e_{20}, e_{00}, e'_{01}, e'_{12}, e_{21}, e_{10}, e_{01})$

Theorem D (Proof)

From 3.

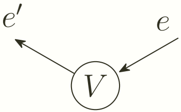
$$\mathrm{init}(e_1) = R = \mathrm{fin}(e_m)$$

$\mathrm{init} = R$	$\mathrm{fin} = R$
e_1	e_m
e_j	$e_j - 1$
$\vdots$	$\vdots$

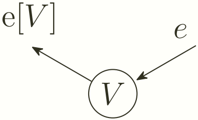
Theorem D (Proof)

If e is an arc not in P, $V = \mathrm{fin}(e)$:

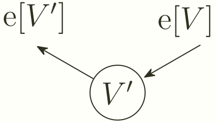
- $\mathcal{G}$ is balanced, so there exists arc e' *not* in P with $\mathrm{init}(e') = V$



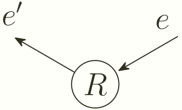
- if $V \neq R$ then $\mathrm{e}[V]$ is not in P (3.)



- Analoguously with $e = \mathrm{e}[V]$

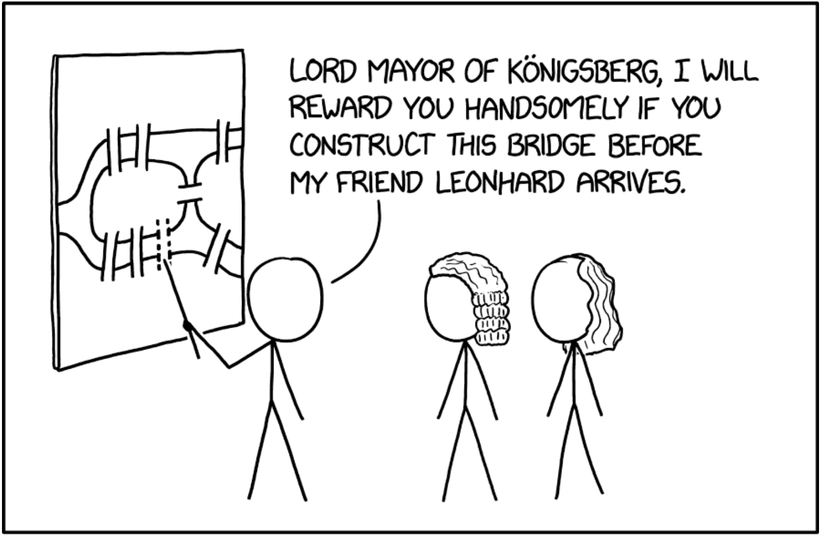


- Until $\mathrm{fin}(e) = R$



$\mathrm{init}(e') = R$ and e' is not in P (↯3.)
$\implies$ e must be in P
$\implies$ P is an eulerian circuit

■



I TRIED TO USE A TIME MACHINE TO CHEAT ON MY ALGORITHMS FINAL BY PREVENTING GRAPH THEORY FROM BEING INVENTED.

xkcd.com/2694