Internalization of a Categorical Automata

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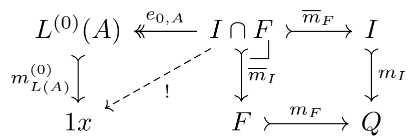
Frank, et. al.[1] define a non-deterministic automaton in a category $\mathscr{C}$ as the quintuplet of objects and subobjects

$$A = (Q \in \mathscr{C}, \Sigma \in \mathscr{C}, m_\delta \colon \delta \rightarrowtail Q \times \Sigma \times Q, m_I \colon I \rightarrowtail Q, m_F \colon F \rightarrowtail Q).$$

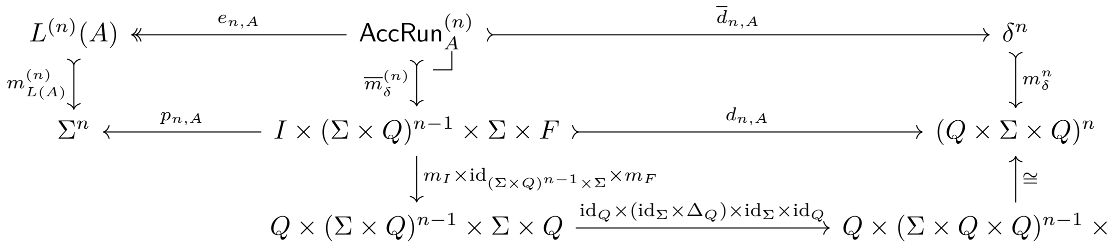
The accepted language of an automaton A is $L(A)$, morally a subobject of $\Sigma^\star$. If $\mathscr{C}$ lacks countable coproducts, which would be necessary to give $\Sigma^\star = \coprod_{n\in\mathbb{N}} \Sigma^n$, it is possible to define $L(A)$ "length-wise", in which case we presuppose a definition of a language as a family of monos

$$L \coloneqq \left(m_n^{(L)} \colon L^{(n)} \rightarrowtail \Sigma^n\right)_{n\in\mathbb{N}}$$

For the empty word, we consider the following diagram, that constructs the pullback $I \cap F$ (designating the initial states that are also accepting), and the image of $I \cap F \longrightarrow 1$,



For non-empty words, the notion of an "accepting run" $\mathsf{AccRun}_A^{(n)}$ of an automaton A for words $\sigma_1 \ldots \sigma_n$ is given by



where $p_{n,A}$ is the projection discarding all states from transitions. The general idea is that an accepting run is a chain of transitions over the subobject of *legal* (δ^n) one-step transitions δ, connecting an *initial state to an accepting state* ($I \times \cdots \times F$). The pullback of $d_{n,A}$ and m^n_δ acts as a generalisation of intersections in **Sets**, resulting in *legal, accepting runs*. By projecting ($p_{n,A}$) out the inputs of type Σ, we get the accepted words.

Categorical Automata in a Topos A topos $\mathscr{E}$ provides all the necessary structure to define a categorical automaton:

- All finite limits, including e.g. to construct the pullback $\mathsf{AccRun}_A^{(n)}$
- Arbitrary subobjects $m \colon S \rightarrowtail A$
- General Epi-Mono Factorisation[2]

and therefore of interest if we can give a

$$m_A^{(n)} \colon \left\{ \sigma_1 \ldots \sigma_n \mid \text{???} \right\} \rightarrowtail \Sigma^n$$

description of a language, such that the above diagram commutes, i.e. the subobject *is* the image of $p_{n,A} \circ \overline{m}^{(n)}_\delta$.

[1] Florian Frank, Stefan Milius, and Henning Urbat. *Positive Data Languages*. 2023. arXiv: 2304.12947 [cs.FL].

[2] Saunders MacLane and Ieke Moerdijk. *Sheaves in geometry and logic: A first introduction to topos theory*. Springer Science & Business Media, 2012, p. 185.

Base case $n = 0$ Here the question is, whether ϵ is part of the language or not. Note here that Q is an internal latice in $\mathscr{E}$.[3]

As $m^{(0)}_{L(A)}$ has 1 as its codomain it is unique up to isomorphism. As $! = m^{(0)}_{L(A)} \circ e_{0,A}$, $e_{0,A}$ is also unique up to iso. The question is therefore, if the domain of $e_{0,A}$ is 0 (when F and I are disjunct subobjects, $F \cap I \cong 0$), in which case $L^{(0)}(A) \cong 0$ as well, meaning that $\epsilon \notin L(A)$.

Alternatively if I and F are *not* disjunct subobjects (meaning there is an initial state that is also accepting, i.e. $\epsilon \in L(A)$), we want

$$L^{(0)}(A) = \eta_{\Sigma^\star}(\epsilon) = \{\epsilon\} = \{\, w \mid w = \epsilon \,\}$$

to hold. We can equivalently characterise the above as

$$L^{(0)}(A) = \{\, \epsilon \mid F \cap I \ncong 0 \,\}.$$

Definition of $L^{(n)}_A$ and $\mathsf{AccRun}^{(n)}_A$ for $n > 0$ The definition of $\mathsf{AccRun}^{(n)}_A$, as the pullback of $m^{(n)}_\delta$ and $d_{n,A}$, gives a straightforward expression in the internal logic of $\mathscr{E}$,

$$\mathsf{AccRun}^{(n)}_A = \left\{ a \;\middle|\; d_{n,A}(\overline{m}^{(n)}_\delta(a)) = m^{(n)}_\delta(\overline{d}_{n,A}(a)) \right\}.$$

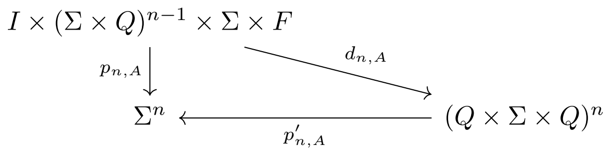
With some foreknowledge, we can describe the image of $m^{(n)}_{L(A)}$ as

$$\mathfrak{L}^{(n)}(A) \coloneqq \left\{ w \;\middle|\; \exists\, r \colon (Q \times \Sigma \times Q)^n .\, w = p'_{n,A}(r) \wedge \mathsf{Accp}^{(n)}_A(r) \right\}$$

where

$$\mathsf{Accp}^{(n)}_A(r) \coloneqq \exists\, t \colon I \times (\Sigma \times Q)^{n-1} \times \Sigma \times F .\, r \in m^n_\delta(\delta^n) \cap d_{n,A}(t)$$

and $p'_{n,A} = \pi^n_2$ (the second projection under a $n$ary-product), such that



commutes, which is easy to see since $d_{n,A}$ preserves all Σ by identities or isomorphisms.

To verify the validity of this description, i.e. that $m^{(n)}_{L(A)}$ is the smallest subobject through which $p_{n,A} \circ \overline{m}^{(n)}_\delta$ can factor, we can take the description of an image in the internal logic

$$\mathrm{Im}(f \colon A \longrightarrow B) = \{\, b \colon B \mid \exists\, a \colon A .\, f(a) = b \,\},$$

and attempt to prove the equivalence of subobjects

$$\vdash \mathrm{Im}\left(p_{n,A} \circ \overline{m}^{(n)}_\delta\right) = \mathfrak{L}^{(n)}(A).$$

It is at this point that we have to provide a concrete and non-recursive definitions of $\overline{m}^{(n)}_\delta$ and $\mathsf{AccRun}^{(n)}_A$. One option is to take $\mathsf{AccRun}^{(n)}_A \coloneqq (Q \times \Sigma \times Q)^n$, such that $\pi_1(\pi_1(\mathsf{AccRun}^{(n)}_A))$ is initial and $\pi_3(\pi_m(\mathsf{AccRun}^{(n)}_A))$ is final, while

$$\pi_3(\pi_i(\mathsf{AccRun}^{(n)}_A)) = \pi_1(\pi_{i+1}(\mathsf{AccRun}^{(n)}_A)) \qquad \forall 1 \leq i < n$$

and have $\overline{m}^{(n)}_\delta$ be the destructing map that requires the first state to be in I and the final state in F, and $\overline{d}_{n,A}$ is a simple injection.

[3] MacLane and Moerdijk, *Sheaves in geometry and logic: A first introduction to topos theory*, p. 198.

Note: Due to the communing property of $p'_{n,A}$, we can equivalently consider $\mathrm{Im}(p'_{n,A} \circ d_{n,A} \circ \overline{m}_\delta^{(n)})$. We can express $\ell := p'_{n,A} \circ d_{n,A} \circ \overline{m}_\delta^{(n)} = \pi_2^n$ (since $d_{n,A} \circ \overline{m}_\delta^{(n)}$ ought to equal $\mathrm{id}_{Q\times\Sigma\times Q}$) in the internal logic as

$$\ell(r) = \{\, w \mid w = \pi_2^n(r) \,\},$$

using which we can define the language as

$$\{\, w \mid \exists\, r.\, \ell(r) = w \,\} = \left\{ w \colon \Sigma^n \;\middle|\; \exists\, r \colon \mathsf{AccRun}_A^{(n)} \subseteq (Q \times \Sigma \times Q)^n .\, \pi_2^n(r) = w \right\} : \mathbf{P}\left(\Sigma^n\right).$$

Equivalence of subobjects With the above we can investigate if the internal descriptions, for $n > 0$:

$$\begin{aligned}
&\vdash && \mathrm{Im}(\pi_2^n) = \mathfrak{L}^{(n)}(A)\\
w\colon \Sigma^n &\vdash && w \in \mathrm{Im}(\pi_2^n) \iff w \in \mathfrak{L}^{(n)}(A)\\
w\colon \Sigma^n &\vdash && w \in \{\, w \mid \exists\, a.\, \pi_2^n(a) = w \,\} \iff w \in \left\{ w \;\middle|\; \exists\, r.\, w = \pi_2^n(r) \wedge \mathsf{Accp}_A^{(n)}(r) \right\}\\
w\colon \Sigma^n &\vdash && \exists\, a \colon \mathsf{AccRun}_A^{(n)} .\, \pi_2^n(a) = w \iff \exists\, r \colon (Q \times \Sigma \times Q)^n .\, w = \pi_2^n(r) \wedge \mathsf{Accp}_A^{(n)}(r)\\
w\colon \Sigma^n &\vdash && \exists\, r \colon (Q \times \Sigma \times Q)^n .\, \pi_2^n(r) = w \wedge \mathsf{Accp}_A^{(n)}(r) \iff \exists\, r \colon (Q \times \Sigma \times Q)^n .\, w = \pi_2^n(r) \wedge \mathsf{Accp}_A^{(n)}(r)
\end{aligned}$$

where the last inference is valid, since $\mathsf{Accp}_A^{(n)}$ is the characteristic morphism that recognises if a $r \colon (Q \times \Sigma \times Q)^n$ is an accepted run.

This fact remains open to proof:

$$\begin{aligned}
&\vdash && \mathsf{AccRun}_A^{(n)} = \left\{ r \colon (Q \times \Sigma \times Q)^n \;\middle|\; \mathsf{Accp}_A^{(n)}(r) \right\}\\
&\vdash && \left\{ a \;\middle|\; d_{n,A}(\overline{m}_\delta^{(n)}(a)) = m_\delta^{(n)}(\bar{d}_{n,A}(a)) \right\} = \left\{ r \colon (Q \times \Sigma \times Q)^n \;\middle|\; \mathsf{Accp}_A^{(n)}(r) \right\}
\end{aligned}$$

The following step intend to address the informality, in that the right-hand side is of the type $\mathsf{AccRun}_A^{(n)}$, while the left-hand side has the super-type $(Q \times \Sigma \times Q)^n$. We do this by "up-casting":

$$\begin{aligned}
&\vdash && \left\{ r \;\middle|\; \exists\, a.\, a = r \wedge d_{n,A}(\overline{m}_\delta^{(n)}(a)) = m_\delta^{(n)}(\bar{d}_{n,A}(a)) \right\} = \left\{ r \colon (Q \times \Sigma \times Q)^n \;\middle|\; \mathsf{Accp}_A^{(n)}(r) \right\}\\
r\colon (Q \times \Sigma \times Q)^n &\vdash && \exists\, a.\, a = r \wedge d_{n,A}(\overline{m}_\delta^{(n)}(a)) = m_\delta^{(n)}(\bar{d}_{n,A}(a)) \iff \exists\, t.\, r \in m_\delta^n(\delta^n) \cap d_{n,A}(t)
\end{aligned}$$

Descending back into irrigorousness, we can disregard the morphisms that serve as injections from subobjects,

$$\begin{aligned}
r\colon (Q \times \Sigma \times Q)^n &\vdash && \exists\, a.\, a = r \wedge d_{n,A}(\overline{m}_\delta^{(n)}(a)) = a \iff \exists\, t.\, r \in \delta^n \cap d_{n,A}(t)\\
r\colon (Q \times \Sigma \times Q)^n &\vdash && \exists\, a \colon \mathsf{AccRun}_A^{(n)} .\, r = d_{n,A}(\overline{m}_\delta^{(n)}(a)) \iff \exists\, t.\, r \in \delta^n \cap d_{n,A}(t)
\end{aligned}$$

Using the property that in a pullback $\mathrm{Pb}(a,b)$, for any $a \colon A \rightarrowtail C$, $b \colon B \rightarrowtail C$ and a fixed $\beta \colon B$ (see below),

$$\gamma \colon C, \beta \colon B \vdash \exists\, \pi \colon \mathrm{Pb}(a,b).\, a(\bar{b}(\pi)) = \gamma \iff \exists\, \alpha \colon A.\, \gamma \in a(\alpha) \cap b(\beta), \tag{†}$$

we can conclude the proof with

$$\begin{aligned}
r\colon (Q \times \Sigma \times Q)^n &\vdash && \exists\, a \colon \mathsf{AccRun}_A^{(n)} .\, r = a \iff \exists\, a \colon \mathsf{AccRun}_A^{(n)} .\, r = d_{n,A}(\overline{m}_\delta^{(n)}(a))\\
r\colon (Q \times \Sigma \times Q)^n &\vdash && \exists\, a.\, r = a \iff \exists\, a.\, r = a \qquad \blacksquare
\end{aligned}$$

Proof of the remaining assumption It remains to argue that (†) is a valid claim. The intuition is pullbacks in $\mathscr{E}$ correspond to intersections in the internal logic of $\mathscr{E}$.

$$\gamma \colon C, \beta \colon B \vdash \quad \exists\, \pi \colon \mathrm{Pb}(a,b).\, a(\bar{b}(\pi)) = \gamma \iff \exists\, \alpha \colon A.\, \gamma \in a(\alpha) \cap b(\beta)$$

We begin by unfolding the definition of "$\cap$",

$$\gamma \colon C, \beta \colon B \vdash \quad \exists\, \pi \colon \mathrm{Pb}(a,b).\, a(\bar{b}(\pi)) = \gamma \iff \exists\, \alpha \colon A.\, \gamma \in \{\, \zeta \colon C \mid \zeta \in a(\alpha) \wedge \zeta \in b(\beta) \,\}$$

and of

$$\mathrm{Pb}(a,b) = \left\{\, \varpi \colon C \;\middle|\; a(\bar{b}(\varpi)) = b(\bar{a}(\varpi)) \,\right\},$$

we can take $\alpha = \bar{b}(\varpi)$ for a $\bar{a}(\varpi) = \beta$. This gives us

$$\gamma \colon C, \beta \colon B \vdash \quad \exists\, \varpi \colon C.\, \bar{a}(\varpi) = \beta \implies a(\bar{b}(\varpi)) = \gamma \iff \gamma \in a(\bar{b}(x)) \wedge \gamma \in b(\beta)$$

TODO

A different approach Take $\sigma_1 \dots \sigma_n \colon \Sigma^n \vdash$, then

$$
\begin{aligned}
&\sigma_1 \dots \sigma_n \in \mathrm{Im}(\pi_2^n) \\
\iff &\sigma_1 \dots \sigma_n \in \left\{ \sigma_1 \dots \sigma_n \;\middle|\; \exists\, a \colon \mathsf{AccRun}_A^{(n)}.\, \sigma_1 \dots \sigma_n = \pi_2^n(a) \right\} \\
\iff &\exists\, a \colon \mathsf{AccRun}_A^{(n)}.\, \sigma_1 \dots \sigma_n = \pi_2^n(a) \\
\iff &\exists\, a \colon \mathsf{AccRun}_A^{(n)}.\, \sigma_1 \dots \sigma_n = \pi_2^n(a) && (\ddagger) \\
\iff &\exists\, r \colon (Q \times \Sigma \times Q)^n.\, \sigma_1 \dots \sigma_n = \pi_2^n(r) \wedge \mathsf{Accp}_A^{(n)}(r) \\
\iff &\sigma_1 \dots \sigma_n \in \left\{ \sigma_1 \dots \sigma_n \;\middle|\; \exists\, r \colon (Q \times \Sigma \times Q)^n.\, \sigma_1 \dots \sigma_n = \pi_2^n(r) \wedge \mathsf{Accp}_A^{(n)}(r) \right\} \\
\iff &\sigma_1 \dots \sigma_n \in \mathfrak{L}^{(n)}(A)
\end{aligned}
$$

where $(\ddagger)$ requires

$$
r \colon (Q \times \Sigma \times Q)^n \vdash \mathsf{Accp}_A^{(n)}(r) \iff \exists\, a \colon \mathsf{AccRun}_A^{(n)}.\, \iota_{(Q\times\Sigma\times Q)^n}(a) = r
$$

to hold. Intuitively this should indicate that $\mathsf{Accp}_A^{(n)}$ is a faithful predicate, recognising if a $r \colon (Q \times \Sigma \times Q)^n$ is part of the intersection of *legal* (m_δ^n) and *accepting* ($d_{n,A}$) runs, i.e. in the pullback of the subobjects m_δ^n and $d_{n,A}$.

Keeping in mind that for a A, δ^n is fixed, this is equivalent to stating that

$$
r \colon (Q\times\Sigma\times Q)^n \vdash \exists\, t \colon I \times \cdots \times F.\, r \in \underbrace{m_\delta^n(\delta^n)}_{\text{legal}} \cap \underbrace{d_{n,A}(t)}_{\text{acc.}} \iff \exists\, \pi \colon \mathrm{Pb}(m_\delta^n, d_{n,A}).\, \overline{d}_{n,A}(\pi) = \delta^n \implies r = \pi
$$

or generally for subobjects $a \colon A \rightarrowtail C$, $b \colon B \rightarrowtail C$ and a fixed $b \colon B$,

$$
\gamma \colon C, \beta \colon B \vdash \exists\, \alpha \colon A.\, \gamma \in a(\alpha) \cap b(\beta) \iff \exists\, \pi \colon \mathrm{Pb}(a, b).\, \overline{a}(\pi) = \beta \implies \iota_C(\pi) = \gamma
$$

(Note to self: Does $\gamma \in a(\alpha) \cap b(\beta)$ even type? I don't think so, since $a(\alpha) \colon C$, and we cannot intersect on an element of C!) So the actual definition of $\mathsf{Accp}_A^{(n)}$ should be

$$
\mathsf{Accp}_A^{(n)}(r) \coloneqq \exists\, a \colon I \times \cdots \times F.\, d_{n,A}(a) = r
$$