An Excordium on Computational Trinitarianism *

Curry-Howard-Lambek Correspondence

As revealed by KALUĐERČIĆ, Philip;

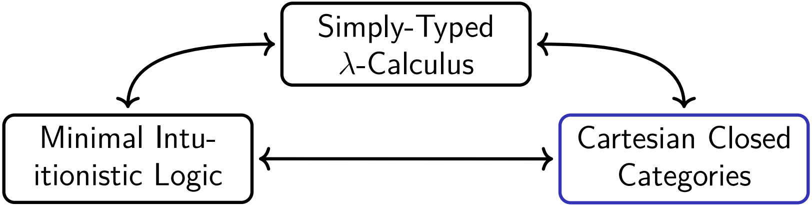
Questions or Complaints? Mail `philip.kaludercic` *at* `fau.de`.

2024-12-19, last typeset December 19, 2024, 19:20

* Available on the WWW: `https://wwwcip.cs.fau.de/~oj14ozun/src+etc/chl.pdf`

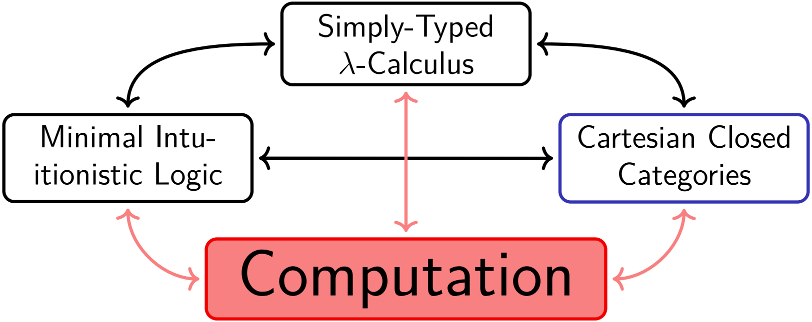
Abstract

Goal: We want to extend the *Logic-Language* correspondance by *Categories*:



Abstract

Goal: We want to extend the *Logic-Language* correspondance by *Categories*:



Each edge represents a “moment” of the essence of computation?

Part I

Yet Another Introduction to Category Theory

Section 1

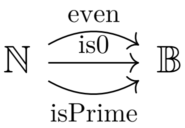
The Definition of a Category

Set-theoretic functions

Posets

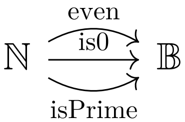
Monoids

Functions "connect" Sets

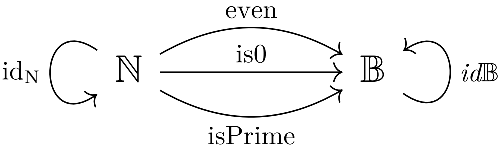


Set-theoretic functions | Posets | Monoids

Functions "connect" Sets



Every set has an "identity function"

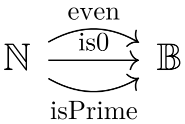


Set-theoretic functions

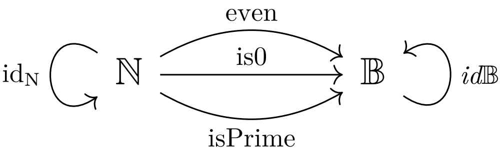
Posets

Monoids

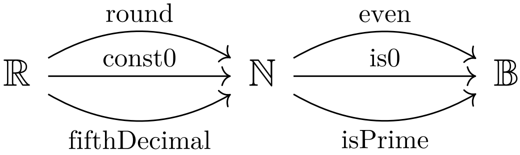
Functions “connect” Sets



Every set has an “identity function”



Functions can be composed (associativley)



Set-theoretic functions

Posets

Monoids

A relation “connect” elements

$$\{a, b\} \subseteq \{a, b, c\}$$

Set-theoretic functions	Posets	Monoids

A relation “connect” elements

$$\{a, b\} \subseteq \{a, b, c\}$$

The relation is reflexive

$$\{a, b\} \subseteq \{a, b\}$$

Set-theoretic functions | Posets | Monoids

A relation "connect" elements

$$\{a, b\} \subseteq \{a, b, c\}$$

The relation is reflexive

$$\{a, b\} \subseteq \{a, b\}$$

The relation is transitive

$$\{a\} \subseteq \{a, b\} \subseteq \{a, b, c\} \quad \text{and}$$

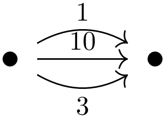
$$\{a, b\} \subseteq \{a, b, c\} \subseteq \{a, b, c, d\}$$

grants

$$\{a\} \subseteq \{a, b\} \subseteq \{a, b, c\} \subseteq \{a, b, c, d\}$$

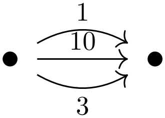
Set-theoretic functions | Posets | Monoids

Elements of a monoid (e.g. $(\mathbb{N}, +, 0)$) "connect"

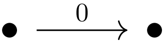


Set-theoretic functions | Posets | Monoids

Elements of a monoid (e.g. $(\mathbb{N}, +, 0)$) "connect"

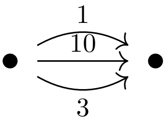


There is a unique neutral "arrow"

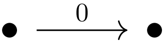


Set-theoretic functions | Posets | Monoids

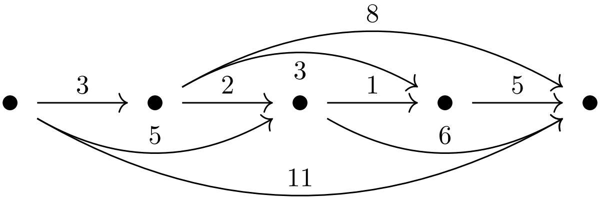
Elements of a monoid (e.g. $(\mathbb{N}, +, 0)$) "connect"



There is a unique neutral "arrow"



All "arrows" are associative



Definition (Category)

A category $\mathscr{C}$ is a

Definition (Category)

A category $\mathscr{C}$ is a collection of objects
$A, B \in \mathsf{Ob}(\mathscr{C})$

Definition (Category)

A category $\mathscr{C}$ is a collection of objects $A, B \in \mathsf{Ob}(\mathscr{C})$ and arrows $\mathrm{Hom}_{\mathscr{C}}(A, B)$ that *associate* objects.

Definition (Category)

A category $\mathscr{C}$ is a collection of objects $A, B \in \mathsf{Ob}(\mathscr{C})$ and arrows $\mathrm{Hom}_{\mathscr{C}}(A, B)$ that *associate* objects. For $A, B, C \in \mathsf{Ob}(\mathscr{C})$, categories must satisfy the properties:

$$\frac{}{\mathrm{id}_A \colon A \longrightarrow A} \text{ (neutral)}$$

Definition (Category)

A category $\mathscr{C}$ is a collection of objects $A, B \in \mathsf{Ob}(\mathscr{C})$ and arrows $\mathrm{Hom}_{\mathscr{C}}(A, B)$ that *associate* objects. For $A, B, C \in \mathsf{Ob}(\mathscr{C})$, categories must satisfy the properties:

$$\frac{}{\mathrm{id}_A \colon A \longrightarrow A} \text{ (neutral)}$$

$$f \colon A \longrightarrow B$$

Definition (Category)

A category $\mathscr{C}$ is a collection of objects $A, B \in \mathsf{Ob}(\mathscr{C})$ and arrows $\mathrm{Hom}_{\mathscr{C}}(A, B)$ that *associate* objects. For $A, B, C \in \mathsf{Ob}(\mathscr{C})$, categories must satisfy the properties:

$$\frac{}{\mathrm{id}_A \colon A \longrightarrow A} \text{ (neutral)}$$

$$f\colon A \longrightarrow B \qquad g\colon B \longrightarrow C$$

Definition (Category)

A category $\mathscr{C}$ is a collection of objects $A, B \in \mathsf{Ob}(\mathscr{C})$ and arrows $\mathrm{Hom}_{\mathscr{C}}(A, B)$ that *associate* objects. For $A, B, C \in \mathsf{Ob}(\mathscr{C})$, categories must satisfy the properties:

$$\frac{}{\mathrm{id}_A \colon A \longrightarrow A}\ \text{(neutral)}$$

$$\frac{f\colon A \longrightarrow B \qquad g\colon B \longrightarrow C}{g \circ f\colon A \longrightarrow C}\ \text{(comp)}$$

Definition (Category)

A category $\mathscr{C}$ is a collection of objects $A, B \in \mathsf{Ob}(\mathscr{C})$ and arrows $\mathrm{Hom}_{\mathscr{C}}(A, B)$ that *associate* objects. For $A, B, C \in \mathsf{Ob}(\mathscr{C})$, categories must satisfy the properties:

$$\frac{}{\mathrm{id}_A \colon A \longrightarrow A} \text{ (neutral)}$$

$$\frac{f\colon A \longrightarrow B \qquad g\colon B \longrightarrow C}{g \circ f\colon A \longrightarrow C} \text{ (comp)}$$

s.t. $\mathrm{id}_B \circ f = f = f \circ \mathrm{id}_A$.

Example (Category of Functions "**Sets**")

Example (Category of Functions "$\mathbf{Sets}$")

$$\mathrm{Ob}\left(\mathbf{Sets}\right) := \text{all sets},$$

Example (Category of Functions "$\mathbf{Sets}$")

$$\mathsf{Ob}\left(\mathbf{Sets}\right) := \text{all sets},$$

$$\mathrm{Hom}_{\mathbf{Sets}}\left(A, B\right) := \text{all functions from } A \text{ to } B.$$

Example (Category of Functions “$\mathbf{Sets}$”)

$$\mathsf{Ob}\,(\mathbf{Sets}) := \text{all sets},$$

$$\mathrm{Hom}_{\mathbf{Sets}}\,(A, B) := \text{all functions from } A \text{ to } B.$$

This is a canonical example of a category. Many other examples restrict $\mathbf{Sets}$ to specific objects and functions ($\mathbf{FinSet}$, $\mathbf{Top}$, $\mathbf{Gra}$, $\mathbf{Grp}$) or generalise it ($\mathbf{Rel}$, $\mathbf{Par}$).

Example (Category of a Poset $(X, \sqsubseteq)$)

Example (Category of a Poset $(X, \sqsubseteq)$)

$$\mathsf{Ob}\left((X, \sqsubseteq)\right) := X,$$

Example (Category of a Poset $(X, \sqsubseteq)$)

$$\mathsf{Ob}\left((X, \sqsubseteq)\right) := X,$$

$$\mathrm{Hom}_{(X,\sqsubseteq)}\left(A, B\right) := \begin{cases} \{*\} & \text{if } A \sqsubset B \\ \{\} & \text{otherwise} \end{cases},$$

for $A, B \in X$.

Example (Category of a Poset $(X, \sqsubseteq)$)

$$\mathsf{Ob}\left((X, \sqsubseteq)\right) := X,$$

$$\mathrm{Hom}_{(X, \sqsubseteq)}\left(A, B\right) := \begin{cases} \{*\} & \text{if } A \sqsubset B \\ \{\} & \text{otherwise} \end{cases},$$

for $A, B \in X$.

This example illustrates that arrows are not always function-*ish*.

Example (Category of a Poset $(M, \cdot, e)$)

Example (Category of a Poset $(M, \cdot, e)$)

$$\mathrm{Ob}\left(\mathbf{Sets}\right) := \{*\},$$

Example (Category of a Poset $(M, \cdot, e)$)

$$\mathsf{Ob}\left(\mathbf{Sets}\right) := \{*\},$$

$$\mathrm{Hom}_{\mathbf{Sets}}\left(*, *\right) := M.$$

Example (Category of a Poset $(M, \cdot, e)$)

$$\mathsf{Ob}\left(\mathbf{Sets}\right) := \{*\},$$

$$\mathrm{Hom}_{\mathbf{Sets}}\left(*, *\right) := M.$$

This example emphasises the “monoidal” nature of categories.

Section 2

Selected Universal Properties of Constructions

Fact (Fun)

Category theory allows us to recognise different settings where objects relate (via arrows) to one another in analogous ways.

Fact (Fun)

Category theory allows us to recognise different settings where objects relate (via arrows) to one another in analogous ways.

Fact (Fun, *continued*)

Of particular interest are constructions that are identified by a unique arrow.

Category $\mathbf{Sets}$

For any A, there exists exactly one function

$$h\colon A \to \{*\}$$

$$a \mapsto *$$

Category Sets

For any A, there exists exactly one function

$$h\colon A \to \{*\}$$
$$a \mapsto *$$

Poset $(X, \sqsubseteq)$

(If there is a top element,) for any $A \in X$, we know that

$$A \sqsubseteq \top$$

must hold. Hence,

$$\mathrm{Hom}\,(A, \top) = \{*\}.$$

Category $\mathbf{Sets}$

For any A, there exists exactly one function

$$h\colon A \to \{*\}$$
$$a \mapsto *$$

Poset $(X, \sqsubseteq)$

(If there is a top element,) for any $A \in X$, we know that

$$A \sqsubseteq \top$$

must hold. Hence,

$$\mathrm{Hom}\,(A, \top) = \{*\}.$$

Monoid $(M, \cdot, e)$

There is only one object, but not only one arrow ($\mathrm{Hom}\,(*, *) = M$).

Category Sets

For any A, there exists exactly one function

$$h\colon A \to \{*\}$$
$$a \mapsto *$$

Poset $(X, \sqsubseteq)$

(If there is a top element,) for any $A \in X$, we know that

$$A \sqsubseteq \top$$

must hold. Hence,

$$\mathrm{Hom}\,(A, \top) = \{*\}.$$

Monoid $(M, \cdot, e)$

There is only one object, but not only one arrow ($\mathrm{Hom}\,(*, *) = M$).

Definition

A category $\mathscr{C}$ with a terminal object $1 \in \mathsf{Ob}\,(\mathscr{C})$ has exactly one arrow

$$!\colon A \longrightarrow T, \qquad |\mathrm{Hom}_{\mathscr{C}}\,(A, T)\,| = 1$$

for every other object $A \in \mathsf{Ob}\,(\mathscr{C})$.

Category **Sets**

Category **Sets**

A product $A \times B$ of two sets has two projections

$$\begin{aligned} \pi_1 \colon A \times B &\to A \\ (a, b) &\mapsto a \\ \pi_2 \colon A \times B &\to B \\ (a, b) &\mapsto b \end{aligned}$$

Category **Sets**

A product $A \times B$ of two sets has two projections

$$\pi_1 \colon A \times B \to A$$
$$(a, b) \mapsto a$$
$$\pi_2 \colon A \times B \to B$$
$$(a, b) \mapsto b$$

Poset $(X, \sqsubseteq)$

Category **Sets**

A product $A \times B$ of two sets has two projections

$$\pi_1 \colon A \times B \to A$$
$$(a, b) \mapsto a$$
$$\pi_2 \colon A \times B \to B$$
$$(a, b) \mapsto b$$

Poset $(X, \sqsubseteq)$

For the meet $A \sqcap B$ we know that

$$A \sqcap B \sqsubseteq A \quad \text{and}$$
$$A \sqcap B \sqsubseteq B$$

must hold.

Category $\mathbf{Sets}$

A product $A \times B$ of two sets has two projections

$$\pi_1 \colon A \times B \to A \quad (a, b) \mapsto a$$
$$\pi_2 \colon A \times B \to B \quad (a, b) \mapsto b$$

Poset $(X, \sqsubseteq)$

For the meet $A \sqcap B$ we know that

$$A \sqcap B \sqsubseteq A \quad \text{and}$$
$$A \sqcap B \sqsubseteq B$$

must hold.

Definition (Preliminary?)

A product "$A \times B$" of two objects $A, B \in \mathrm{Ob}(\mathscr{C})$ has two arrows $A \times B \longrightarrow A$ and $A \times B \longrightarrow B$.

Definition

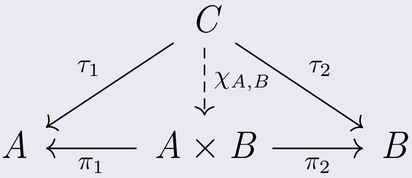
The product "$A \times B$" of two objects $A, B \in \mathsf{Ob}(\mathscr{C})$ has two arrows $\pi_1\colon A \times B \longrightarrow A$ and $\pi_2\colon A \times B \longrightarrow B$,

Definition

The product "$A \times B$" of two objects $A, B \in \mathsf{Ob}(\mathscr{C})$ has two arrows $\pi_1 \colon A \times B \longrightarrow A$ and $\pi_2 \colon A \times B \longrightarrow B$, such that for any other object $C \in \mathsf{Ob}(\mathscr{C})$ with $\tau_1 \colon C \longrightarrow A$ and $\tau_2 \colon C \longrightarrow B$,

Definition

The product "$A \times B$" of two objects $A, B \in \mathsf{Ob}(\mathscr{C})$ has two arrows $\pi_1 \colon A \times B \longrightarrow A$ and $\pi_2 \colon A \times B \longrightarrow B$, such that for any other object $C \in \mathsf{Ob}(\mathscr{C})$ with $\tau_1 \colon C \longrightarrow A$ and $\tau_2 \colon C \longrightarrow B$,



Example (In $\mathbf{Sets}$ (I))

Why is $X \times (Y \times Z)$ not the product of X and Z?

Example (In $\mathbf{Sets}$ (I))

Why is $X \times (Y \times Z)$ not the product of X and Z?
We can define

$$\tau_1 = \pi_1$$
$$\tau_2 = \pi_2 \circ \pi_2$$

Example (In $\mathbf{Sets}$ (I))

Why is $X \times (Y \times Z)$ not the product of X and Z? We can define

$$\tau_1 = \pi_1$$
$$\tau_2 = \pi_2 \circ \pi_2$$

But there *is* a unique

$$h(x, y, z) = (x, z),$$

while there need not be a $g \colon X \times Z \longrightarrow X \times Y \times Z$

Example (In $\mathbf{Sets}$ (I))

Why is $X \times (Y \times Z)$ not the product of X and Z? We can define

$$\tau_1 = \pi_1$$
$$\tau_2 = \pi_2 \circ \pi_2$$

But there *is* a unique

$$h(x, y, z) = (x, z),$$

while there need not be a $g\colon X \times Z \longrightarrow X \times Y \times Z$ — let alone unique!

Example (In **Sets** (I))

Why is $X \times (Y \times Z)$ not the product of X and Z? We can define

$$\tau_1 = \pi_1$$
$$\tau_2 = \pi_2 \circ \pi_2$$

But there *is* a unique

$$h(x, y, z) = (x, z),$$

while there need not be a $g \colon X \times Z \longrightarrow X \times Y \times Z$ — let alone unique! $X \times Z$ is a more sufficient fit than $X \times Y \times Z$.

Example (In $\mathbf{Sets}$ (II))

Is $Z \times X$ a product of X and Z?

Example (In $\mathbf{Sets}$ (II))

Is $Z \times X$ a product of X and Z? We can define

$$\tau_1 = \pi_2 \qquad \tau_2 = \pi_1$$

Example (In $\mathbf{Sets}$ (II))

Is $Z \times X$ a product of X and Z? We can define

$$\tau_1 = \pi_2 \qquad \tau_2 = \pi_1$$

There is both a unique $h\colon Z \times X \longrightarrow X \times Z$ as

$$(z, x) \mapsto (x, z)$$

and a unique $h^{-1}\colon X \times Z \longrightarrow Z \times X$ as

$$(x, z) \mapsto (z, x).$$

Example (In $\mathbf{Sets}$ (II))

Is $Z \times X$ a product of X and Z? We can define

$$\tau_1 = \pi_2 \qquad \tau_2 = \pi_1$$

There is both a unique $h\colon Z \times X \longrightarrow X \times Z$ as

$$(z, x) \mapsto (x, z)$$

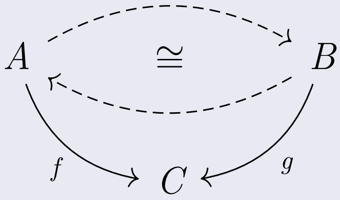
and a unique $h^{-1}\colon X \times Z \longrightarrow Z \times X$ as

$$(x, z) \mapsto (z, x).$$

Both are equally well fit and are mutually correspond to one another.

Fact (...up to isomorphism)

When thinking categorically and considering the relations of objects over the contents of the objects, we handle objects within a equivalence class of "isomorphisms".



Definition

Given $\mathscr{C}$ with all products $- \times Y$,

Definition

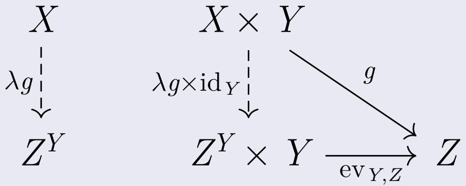
Given $\mathscr{C}$ with all products $- \times Y$, an exponential object Z^Y is such that for any $g\colon X \times Y \longrightarrow Z$,

Definition

Given $\mathscr{C}$ with all products $- \times Y$, an exponential object Z^Y is such that for any $g\colon X \times Y \longrightarrow Z$, there is a unique transpose (sometimes “adjoint”) $\lambda g\colon X \longrightarrow Z^Y$

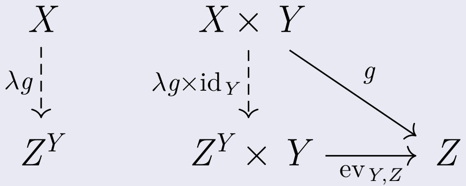
Definition

Given $\mathscr{C}$ with all products $- \times Y$, an exponential object Z^Y is such that for any $g\colon X \times Y \longrightarrow Z$, there is a unique transpose (sometimes "adjoint") $\lambda g\colon X \longrightarrow Z^Y$



Definition

Given $\mathscr{C}$ with all products $- \times Y$, an exponential object Z^Y is such that for any $g\colon X \times Y \longrightarrow Z$, there is a unique transpose (sometimes "adjoint") $\lambda g\colon X \longrightarrow Z^Y$



Example

In $\mathbf{Sets}$ B^A is represents all functions from A to B.

The before can be expressed as the equation:

$$\mathrm{Hom}_{\mathscr{C}}\left(X \times\ Y, Z\right) \cong \mathrm{Hom}_{\mathscr{C}}\left(X, Z^{Y}\right)$$

The before can be expressed as the equation:

$$\mathrm{Hom}_{\mathscr{C}}(X \times Y, Z) \cong \mathrm{Hom}_{\mathscr{C}}\left(X, Z^{Y}\right)$$

Do you recognise this from somewhere?

Fact

We can construct a category H of a Heyting Algebra analogously to the category of a poset.

Fact

We can construct a category H of a Heyting Algebra analogously to the category of a poset.

Example

The exponential object in Heyting Algebra following from the above, corresponds to the well-known definition of implication:

$$a \sqcap b \sqsubseteq c \iff a \sqsubseteq b^c$$

Definition

A Cartesian Closed Category has...

Definition

A Cartesian Closed Category has...

- A terminal object 1,

Definition

A Cartesian Closed Category has...

- A terminal object 1,
- *All* products $A \times B$,

Definition

A Cartesian Closed Category has...

- A terminal object 1,
- *All* products $A \times B$,
- *All* exponentials B^A.

Definition

A Cartesian Closed Category has...

- A terminal object 1,
- *All* products $A \times B$,
- *All* exponentials B^A.

Example

Categories that satisfy these properties include $\mathbf{Sets}$, categories of Heyting Algebras.

Definition

A Cartesian Closed Category has...

- A terminal object 1,
- *All* products $A \times B$,
- *All* exponentials B^A.

Example

Categories that satisfy these properties include $\mathbf{Sets}$, categories of Heyting Algebras.

Now what does all of this have to do with the λ-Calculus or (positive/minimal) intuitionist logic?

Before continuing; What we have omitted?

- Duality,

Before continuing; What we have omitted?

- Duality,
- Isomorphisms,

Before continuing; What we have omitted?

- Duality,
- Isomorphisms,
- Mono- and Epimorphisms,

Before continuing; What we have omitted?

- Duality,
- Isomorphisms,
- Mono- and Epimorphisms,
- Limits,

Before continuing; What we have omitted?

- Duality,
- Isomorphisms,
- Mono- and Epimorphisms,
- Limits,
- Initial Objects, Pushouts, Pullbacks, (Co-)Equalisers,

Before continuing; What we have omitted?

- Duality,
- Isomorphisms,
- Mono- and Epimorphisms,
- Limits,
- Initial Objects, Pushouts, Pullbacks, (Co-)Equalisers,
- Functors,

Before continuing; What we have omitted?

- Duality,
- Isomorphisms,
- Mono- and Epimorphisms,
- Limits,
- Initial Objects, Pushouts, Pullbacks, (Co-)Equalisers,
- Functors,
- Natural Transformations,

Before continuing; What we have omitted?

- Duality,
- Isomorphisms,
- Mono- and Epimorphisms,
- Limits,
- Initial Objects, Pushouts, Pullbacks, (Co-)Equalisers,
- Functors,
- Natural Transformations,
- Adjunctions, units, counits,

Before continuing; What we have omitted?

- Duality,
- Isomorphisms,
- Mono- and Epimorphisms,
- Limits,
- Initial Objects, Pushouts, Pullbacks, (Co-)Equalisers,
- Functors,
- Natural Transformations,
- Adjunctions, units, counits,
- Yoneda Lemma, Embeddings, representable objects,

Before continuing; What we have omitted?

- Duality,
- Isomorphisms,
- Mono- and Epimorphisms,
- Limits,
- Initial Objects, Pushouts, Pullbacks, (Co-)Equalisers,
- Functors,
- Natural Transformations,
- Adjunctions, units, counits,
- Yoneda Lemma, Embeddings, representable objects,
- Kan Extensions,

Before continuing; What we have omitted?

- Duality,
- Isomorphisms,
- Mono- and Epimorphisms,
- Limits,
- Initial Objects, Pushouts, Pullbacks, (Co-)Equalisers,
- Functors,
- Natural Transformations,
- Adjunctions, units, counits,
- Yoneda Lemma, Embeddings, representable objects,
- Kan Extensions,
- Twisted Generalized Cohomology in Linear Homotopy Type Theory, ...

Part II

Equational Theories and λ-Theories

Definition

Equational Theories presents the rules of a formal system with equality as the primary predicate.

Definition

Equational Theories presents the rules of a formal system with equality as the primary predicate.

For example, we expect the following to hold for a equivalence relation:

Definition

Equational Theories presents the rules of a formal system with equality as the primary predicate.

For example, we expect the following to hold for a equivalence relation:

$$\frac{}{A = A}\ (\mathsf{refl})$$

Definition

Equational Theories presents the rules of a formal system with equality as the primary predicate.

For example, we expect the following to hold for a equivalence relation:

$$\frac{}{A = A}\ (\mathsf{refl})$$

$$\frac{B = A}{A = B}\ (\mathsf{sym})$$

Definition

Equational Theories presents the rules of a formal system with equality as the primary predicate.

For example, we expect the following to hold for a equivalence relation:

$$\frac{}{A = A}\ (\mathsf{refl})$$

$$\frac{B = A}{A = B}\ (\mathsf{sym})$$

$$\frac{A = B \qquad B = C}{A = C}\ (\mathsf{trans})$$

Definition

A (simply typed) λ Theory is an equational theory that describes what a equivalence relation between λ-terms should ensure.

We write

$$\Gamma \vdash s = t \colon A$$

to state that $s \colon A$ and $t \colon A$ are equal in the same context Γ.

Definition

A (simply typed) λ Theory is an equational theory that describes what a equivalence relation between λ-terms should ensure.

Definition

A (simply typed) λ Theory is an equational theory that describes what a equivalence relation between λ-terms should ensure.

$$\frac{\Gamma \vdash s = t \colon A \qquad \Gamma, x \colon A \vdash u = v \colon B}{\Gamma \vdash u\ [x \mapsto s] = v\ [x \mapsto t] : B}\ \text{(subst)}$$

Definition

A (simply typed) λ Theory is an equational theory that describes what a equivalence relation between λ-terms should ensure.

$$\frac{\Gamma \vdash s = t \colon A \qquad \Gamma, x \colon A \vdash u = v \colon B}{\Gamma \vdash u\ [x \mapsto s] = v\ [x \mapsto t] : B}\ \text{(subst)}$$

$$\frac{\Gamma \vdash s = t \colon A \to B \qquad \Gamma \vdash u = v \colon A}{\Gamma \vdash su = tv \colon B}\ \text{(app)}$$

Definition

A (simply typed) λ Theory is an equational theory that describes what a equivalence relation between λ-terms should ensure.

$$\frac{\Gamma \vdash s = t\colon A \qquad \Gamma, x\colon A \vdash u = v\colon B}{\Gamma \vdash u\ [x \mapsto s] = v\ [x \mapsto t] : B}\ \text{(subst)}$$

$$\frac{\Gamma \vdash s = t\colon A \to B \qquad \Gamma \vdash u = v\colon A}{\Gamma \vdash su = tv\colon B}\ \text{(app)}$$

$$\frac{\Gamma, x\colon A \vdash t = s\colon B}{\Gamma \vdash \lambda\, x.\, t = \lambda\, x.\, s\colon A \to B}\ \text{(abstr)}$$

Definition

A (simply typed) λ Theory is an equational theory that describes what a equivalence relation between λ-terms should ensure.

$$\frac{\Gamma \vdash s = t\colon A \qquad \Gamma, x\colon A \vdash u = v\colon B}{\Gamma \vdash u\ [x \mapsto s] = v\ [x \mapsto t] : B}\ \text{(subst)}$$

$$\frac{\Gamma \vdash s = t\colon A \to B \qquad \Gamma \vdash u = v\colon A}{\Gamma \vdash su = tv\colon B}\ \text{(app)}$$

$$\frac{\Gamma, x\colon A \vdash t = s\colon B}{\Gamma \vdash \lambda\, x.\, t = \lambda\, x.\, s\colon A \to B}\ \text{(abstr)}$$

$$\frac{}{\Gamma \vdash \lambda\, x.\, ts = t\ [x \mapsto s] : B}\ (\beta)$$

Definition

A (simply typed) λ Theory is an equational theory that describes what a equivalence relation between λ-terms should ensure.

$$\frac{\Gamma \vdash s = t\colon A \qquad \Gamma, x\colon A \vdash u = v\colon B}{\Gamma \vdash u\ [x \mapsto s] = v\ [x \mapsto t] : B}\ \text{(subst)}$$

$$\frac{\Gamma \vdash s = t\colon A \to B \qquad \Gamma \vdash u = v\colon A}{\Gamma \vdash su = tv\colon B}\ \text{(app)}$$

$$\frac{\Gamma, x\colon A \vdash t = s\colon B}{\Gamma \vdash \lambda\, x.\, t = \lambda\, x.\, s\colon A \to B}\ \text{(abstr)}$$

$$\frac{}{\Gamma \vdash \lambda\, x.\, ts = t\ [x \mapsto s] : B}\ (\beta)$$

...rules for product and unit types ...

Part III

Qu'est-ce qui correspond à quoi ?

(What does correspond to what?)

Fact

The general idea of the correspondence is...

$$\mathit{Types} \iff \mathit{Objects}$$

$$\mathit{Terms} \iff \mathit{Arrows}$$

Fact

The general idea of the correspondence is...

$$\textit{Types} \iff \textit{Objects} \; (\iff \textit{Propositions})$$

$$\textit{Terms} \iff \textit{Arrows} \; (\iff \textit{Proofs})$$

Fact

The general idea of the correspondence is...

$$\textit{Types} \iff \textit{Objects} \; (\iff \textit{Propositions})$$

$$\textit{Terms} \iff \textit{Arrows} \; (\iff \textit{Proofs})$$

So demonstrating the existence of a arrow is the same "moral act" as a constructive proof of a proposition or inhabiting a type.

Fact (Intermission)

Any CCC promises the (conditional) existence following arrows:

Fact (Intermission)

Any CCC promises the (conditional) existence following arrows:

1. *For any object A, we have* $! \colon A \longrightarrow 1$

Fact (Intermission)

Any CCC promises the (conditional) existence following arrows:

1. *For any object* A, *we have* $!\colon A \longrightarrow 1$
2. *For any objects* A, B *and a sufficient* C, *we have* $\chi_{A,B}\colon C \longrightarrow A \times B$.

Fact (Intermission)

Any CCC promises the (conditional) existence following arrows:

1. *For any object A, we have $!\colon A \longrightarrow 1$*
2. *For any objects A, B and a sufficient C, we have $\chi_{A,B}\colon C \longrightarrow A \times B$.*
3. *For any $A \times B$ we have $\pi_1\colon A \times B \longrightarrow A$ and $\pi_2\colon A \times B \longrightarrow B$.*

Fact (Intermission)

Any CCC promises the (conditional) existence following arrows:

1. *For any object* A*, we have* $!\colon A \longrightarrow 1$
2. *For any objects* A, B *and a sufficient* C*, we have* $\chi_{A,B}\colon C \longrightarrow A \times B$*.*
3. *For any* $A \times B$ *we have* $\pi_1\colon A \times B \longrightarrow A$ *and* $\pi_2\colon A \times B \longrightarrow B$*.*
4. *For any arrow* $g\colon A \times B \longrightarrow C$ *we have a corresponding* $\lambda g\colon A \longrightarrow C^B$ *(and vice versa).*

Fact (Intermission)

Any CCC promises the (conditional) existence following arrows:

1. *For any object* A*, we have* $!\colon A \longrightarrow 1$
2. *For any objects* A, B *and a sufficient* C*, we have* $\chi_{A,B}\colon C \longrightarrow A \times B$.
3. *For any* $A \times B$ *we have* $\pi_1\colon A \times B \longrightarrow A$ *and* $\pi_2\colon A \times B \longrightarrow B$.
4. *For any arrow* $g\colon A \times B \longrightarrow C$ *we have a corresponding* $\lambda g\colon A \longrightarrow C^B$ *(and vice versa).*
5. *For any* A *and* B^A *we have a* $\mathrm{ev}_{A,B}\colon A \times B^A \longrightarrow B$

λ-Calculus CCC

λ-Calculus

Application of terms:

$$s\colon A \to B, t\colon A \vdash st\colon B$$

CCC

The evaluation arrow:

$$\mathrm{ev}_{A,B}\colon B^A \times A \longrightarrow B$$

λ-Calculus

Application of terms:

$$s\colon A \to B, t\colon A \vdash st\colon B$$

Lambda Abstractions:

$$\vdash \lambda\, x.\, t\colon A \to B$$

CCC

The evaluation arrow:

$$\mathrm{ev}_{A,B}\colon B^A \times A \longrightarrow B$$

The global element/point:

$$\lambda g\colon 1 \longrightarrow B^A$$

λ-Calculus	CCC
Application of terms:	The evaluation arrow:
$s\colon A \to B, t\colon A \vdash st\colon B$	$\mathrm{ev}_{A,B}\colon B^A \times A \longrightarrow B$
Lambda Abstractions:	The global element/point:
$\vdash \lambda\, x.\, t\colon A \to B$	$\lambda g\colon 1 \longrightarrow B^A$
...or by deduction theorem	...or by transposition
$x\colon A \vdash t\colon B$	$g\colon 1 \times A \cong A \longrightarrow B$

λ-Calculus	CCC
Application of terms:	The evaluation arrow:
$s\colon A \to B, t\colon A \vdash st\colon B$	$\mathrm{ev}_{A,B}\colon B^A \times A \longrightarrow B$
Lambda Abstractions:	The global element/point:
$\vdash \lambda\, x.\, t\colon A \to B$	$\lambda g\colon 1 \longrightarrow B^A$
...or by deduction theorem	...or by transposition
$x\colon A \vdash t\colon B$	$g\colon 1 \times A \cong A \longrightarrow B$

With the obvious correspondences between product types and categorical products ($\texttt{fst} \approx \pi_1$, $\texttt{snd} \approx \pi_2$, ...).

We can inhabit type

$$(A \to B) \times ((A \to B) \to C) \to A \to B \times C$$

by the term

We can inhabit type

$$(A \to B) \times ((A \to B) \to C) \to A \to B \times C$$

by the term

$$\lambda\, p.\, \lambda\, a.\, ((\texttt{fst}\ p)a, (\texttt{snd}\ p)(\texttt{fst}\ p))$$

Proof.

Obvious, duh. □

We can prove the intuitionistic proposition is satisfiable

$$(A \to B) \wedge ((A \to B) \to C) \to A \to B \wedge C$$

Proof.

... by constructing the proof tree

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\overline{(A \to B) \wedge ((A \to B) \to C)}}{A \to B}(\wedge_{E1}) \quad \overline{A}}{B} \qquad \dfrac{\dfrac{\overline{(A \to B) \wedge ((A \to B) \to C)}}{(A \to B) \to C}(\wedge_{E2}) \quad \dfrac{\overline{(A \to B) \wedge ((A \to B) \to C)}}{A \to B}(\wedge_{E1})}{C}}{B \wedge C}}{A \to B \wedge C}(\to_I)}{(A \to B) \wedge ((A \to B) \to C) \to A \to B \wedge C}(\to_I)$$

□

We can demonstrate that the following arrow exists

$$1 \longrightarrow (B \times C)^{A^{B^A \times C^{B^A}}}$$

We can demonstrate that the following arrow exists

$$1 \xrightarrow{\lambda\lambda f} (B \times C)^{A^{B^A \times C^{B^A}}}$$

Proof.

... as it is the transpose of

$$A \times B^A \times C^{B^A} \xrightarrow{f} (B \times C)$$

We can demonstrate that the following arrow exists

$$1 \xrightarrow{\lambda\lambda f} (B \times C)^{A^{B^A \times C^{B^A}}}$$

Proof.

... as it is the transpose of

$$A \times B^A \times C^{B^A} \xrightarrow{f} (B \times C)$$

that in turn is given by

$$f := \left\langle \mathrm{ev}_{A,B} \circ \langle \pi_2, \pi_1 \rangle, \mathrm{ev}_{B^A,C} \circ \langle \pi_3, \pi_2 \rangle \right\rangle \qquad \square$$

Counter-Example: We can intuit that the we can't satisfy $\vdash (A \to B) \to (B \to A)$.

Counter-Example: We can intuit that the we can't satisfy $\vdash (A \to B) \to (B \to A)$.

Proof .

$$\rightsquigarrow 1 \xrightarrow{\quad f \quad} \left(A^B\right)^{B^A}$$

Counter-Example: We can intuit that the we can't satisfy $\vdash (A \to B) \to (B \to A)$.

Proof .

$$\rightsquigarrow 1 \xrightarrow{f} \left(A^B\right)^{B^A}$$

$$A^B \xrightarrow{\lambda f} B^A$$

Counter-Example: We can intuit that the we can't satisfy $\vdash (A \to B) \to (B \to A)$.

Proof .

$$\rightsquigarrow 1 \xrightarrow{f} \left(A^B\right)^{B^A}$$

$$A^B \xrightarrow{\lambda f} B^A$$

$$A \times A^B \xrightarrow{\lambda\lambda f} B$$

Counter-Example: We can intuit that the we can't satisfy $\vdash (A \to B) \to (B \to A)$.

Proof "by exhaustion".

$$\rightsquigarrow 1 \xrightarrow{f} \left(A^B\right)^{B^A}$$

$$A^B \xrightarrow{\lambda f} B^A$$

$$A \times A^B \xrightarrow{\lambda\lambda f} B$$

doesn't leave us with any means to construct arrow in an arbitrary CCC (recall the "Intermission"). □

Essentially we are giving a categorical interpretation of λ terms:

$$\begin{aligned}
[\![A \times B]\!] &= [\![A]\!] \times [\![B]\!] \\
[\![A \to B]\!] &= [\![B]\!]^{[\![A]\!]} \\
[\![a\colon A]\!] &= 1 \longrightarrow [\![A]\!] \\
[\![\Gamma \vdash t\colon B]\!] &= [\![\Gamma]\!] \longrightarrow [\![t\colon B]\!] \\
&\vdots
\end{aligned}$$

Essentially we are giving a categorical interpretation of λ terms:

$$\begin{aligned} [\![A \times B]\!] &= [\![A]\!] \times [\![B]\!] \\ [\![A \to B]\!] &= [\![B]\!]^{[\![A]\!]} \\ [\![a\colon A]\!] &= 1 \longrightarrow [\![A]\!] \\ [\![\Gamma \vdash t\colon B]\!] &= [\![\Gamma]\!] \longrightarrow [\![t\colon B]\!] \\ &\vdots \end{aligned}$$

To ensure that the interpretation is sound and complete we need to prove that the rules of the λ theory $\mathbb{T}$ coincide with arrow-equality.

Definition

We say that λ-Calculus is the internal language of Cartesian Closed Categories.

Part IV

Pour aller plus loin

(To go further)

Fact

The more "structure" a category has, the more interesting the internal logic[†]:

[†]See `https://ncatlab.org/nlab/show/internal+logic`

Fact

The more "structure" a category has, the more interesting the internal logic[†]:

- *A topos (more on that in a moment) corresponds to finitist, intuitionistic higher-order logic*

[†] See `https://ncatlab.org/nlab/show/internal+logic`

Fact

The more "structure" a category has, the more interesting the internal logic[†]*:*

- *A topos (more on that in a moment) corresponds to finitist, intuitionistic higher-order logic*
- *A boolean topos (ie. with well-behaved complements) corresponds to classical higher-order logic*

[†] See https://ncatlab.org/nlab/show/internal+logic

Fact

The more "structure" a category has, the more interesting the internal logic[†]*:*

- *A topos (more on that in a moment) corresponds to finitist, intuitionistic higher-order logic*
- *A boolean topos (ie. with well-behaved complements) corresponds to classical higher-order logic*
- *A symmetric monoidal category (generalisation of CCC) corresponds to linear logic*

[†] See `https://ncatlab.org/nlab/show/internal+logic`

Definition

A (elementary) topos $\mathscr{E}$ is a category is a CCC with "all finite limits" and a subobject classifier Ω.

Definition

A (elementary) topos $\mathscr{E}$ is a category is a CCC with "all finite limits" and a subobject classifier Ω.

Fact

The internal language of a topos allows us to reason pointwise about (sub-)objects and even use set-notation:

$$\{\, a \colon A \mid \phi(a) \to \neg\psi(a, a) \,\} : \Omega^A$$

Definition

A (elementary) topos $\mathscr{E}$ is a category is a CCC with "all finite limits" and a subobject classifier Ω.

Fact

The internal language of a topos allows us to reason pointwise about (sub-)objects and even use set-notation:

$$\{\, a\colon A \mid \phi(a) \to \neg\psi(a,a) \,\} : \Omega^A$$

But going into this in detail would be too technical...

Definition

A (elementary) topos $\mathscr{E}$ is a category is a CCC with "all finite limits" and a subobject classifier Ω.

Fact

The internal language of a topos allows us to reason pointwise about (sub-)objects and even use set-notation:

$$\{\, a\colon A \mid \phi(a) \to \neg\psi(a,a) \,\} : \Omega^A$$

But going into this in detail would be too technical... Come back again to my master's presentation next month.

Further Reading and Sources I

Recommended Reading on Category Theory

- https://arxiv.org/pdf/1612.09375
- Book "Categories for the working mathematician" (Mac Lane)
- Book " Basic Category Theory for Computer Scientists" (Pierce)
- https://web.archive.org/web/20230301160845/https://people.math.harvard.edu/~mazur/preprints/when_is_one.pdf

Recommended Reading on Categorical Logic

- https://awodey.github.io/catlog/notes/ (WIP)

Further Reading and Sources II

- `https://plato.stanford.edu/entries/lambda-calculus/#LThe`
- `https://golem.ph.utexas.edu/category/2006/08/cartesian_closed_categories_an_1.html`
- Book “Introduction to Higher Order Categorical Logic” (Lambek)
- Book “The Lambda Calculus, its Syntax and Semantics” (Barendregt)
- Book “Topoi: The Categorial Analysis of Logic” (Goldblatt)

Related and more complicated concepts

Further Reading and Sources III

- https://math.ucr.edu/home/baez/rosetta.pdf#page=66
- Book “Elementary Categories, Elementary Toposes” (McLarty)
- Book “Sketches of an Elephant” (Johnstone)
- Book “Sheaves and Geometry in Logic” (Mac Lane)
- Book “Handbook of Categorical Algebra” (Borceux), specifically Volume 3

A possible first step in the research program is 1700 doctoral theses called “A Correspondence between x and Church’s -notation.”.

— *A popular joke*