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Semantics of Categorical Nondeterministic Automata in a Topos

Master Thesis in Computer Science

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Erlangen, November 11, 2024

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Contents

1 Introduction

The topic is topos.

1.1 An Overview

It is a well known fact, that an introduction has to be written last. This includes, but is not limited to, an overview (such as this section).

2 Technical Prolegomena

2.1 A Brief Introduction to Topos Theory

A quick and informal definition of an *(elementary) Topos* is a category with sufficient structure to model at least intuitionistic set theory. This fact allows us to reason soundly about categorical statements in the *language* of set theory.

As an intermediate step towards understanding what the colloquial notion "structure" of a category designates, recall the following standard definition:

Definition 2.1 (Cartesian Closed)**.** A cartesian closed category (CCC) is a category $\mathscr{C}$ with

1. a terminal object 1,
2. all binary products $A \times B$ for $A, B \in \mathsf{Ob}(\mathscr{C})$,
3. all exponentials B^A for $A, B \in \mathsf{Ob}(\mathscr{C})$.

Categories that exhibit the sufficient properties to be CCCs include sets **Sets** or finite sets **FinSet**, the category of G-sets of a group G, the category of presheafs $\mathbf{Sets}^{\mathscr{C}}$ and the category of CPOs. A Counterexample is the general category of topological spaces **Top**, as this does not have *all* exponentials.

This section *formally* introduces and frames the definitions necessary in the subsequent chapters. At the same time, it should also serve as a general introduction to topos theory, for any reader interested in the "set-like" and logical aspects of the field. For further general literature on the study of toposes, consult "Sheaves in Geometry and Logic" by Mac Lane and Moerdijk [MM12], "Elementary Categories, Elementary Toposes" by McLarty [McL92] or for more advanced details Johnstone's "Topos Theory"[Joh14] along with the "Sketches of an Elephant" [Joh02]. Shorter introductions worthy of recommendation are Tom Leinster's "An informal introduction to topos theory" [Lei10] or John Baez's "Topos Theory in a Nutshell" [Bae21].

2.1.1 Elementary Topos

The notion of an "elementary topos" is due to Lawvere and Tierney, trying to unify aspects from logic and algebraic geometry [MR12, Sec. 3.1]. The conventional definition is as follows:

Definition 2.2 (Elementary Topos)**.** A category $\mathscr{E}$ is an (elementary) topos if

1. $\mathscr{E}$ has all finite limits;
2. $\mathscr{E}$ is cartesian closed, as in Definition 2.1; and
3. $\mathscr{E}$ has a subobject classifier, i.e. an object Ω and a morphism $1 \rightarrowtail \Omega$ (called "true") such that for each monomorphism $m\colon S \rightarrowtail B$ in $\mathscr{E}$, there is a unique characteristic morphism $\phi_m\colon B \longrightarrow \Omega$ (the classifying map of m) making the following diagram a pullback-square:

$$\begin{array}{ccc} S & \xrightarrow{\;!\;} & 1 \\ {\scriptstyle m}\big\downarrow & \lrcorner & \big\downarrow{\scriptstyle \text{true}} \\ B & \xrightarrow{\;\phi_m\;} & \Omega \end{array} \tag{2.1}$$

We call a morphism $e\colon 1 \rightarrowtail A$, for some object $A \in \mathsf{Ob}(\mathscr{E})$ a "global element" of A. For example, the aforementioned $\mathrm{true}\colon 1 \longrightarrow \Omega$ is a global element denoting the truth element of a subobject classifier.

For the sake of consistency, we will denote toposes by $\mathscr{E}$ or $\mathscr{J}$, and arbitrary categories by $\mathscr{C}$ and $\mathscr{D}$.

As mentioned above, a topos has sufficient "structure" to express concepts from set theory. A construct of particular relevance to this thesis, powersets, have a topos-theoretical generalisation:

Definition 2.3 (Power Object)**.** The power object $\mathbf{P}\,B$ of any object B is such that for an arbitrary $f\colon B \times A \longrightarrow \Omega$, there exists a unique $g\colon A \longrightarrow \mathbf{P}\,B$ such that the following commutes:

$$\begin{array}{ccc} A & & B \times A & & \\ {\scriptstyle g}\,\downarrow & & \downarrow{\scriptstyle \mathrm{id}_B \times g} & \searrow^{f} & \\ \mathbf{P}\,B & & B \times \mathbf{P}\,B & \xrightarrow[\in_B]{} & \Omega \end{array} \tag{2.2}$$

In a category with exponentials and a subobject classifier Ω (such as a topos), the power object $\mathbf{P}\,B$ is isomorphic to Ω^B.

Definition 2.4 (Subobject)**.** For two arbitrary monos $m\colon S \rightarrowtail B$ and $m'\colon S' \rightarrowtail B$ with the same codomain B, the existence of a morphism $f\colon S \longrightarrow S'$ such that $m' \circ f = m$ induces a partial order.

A subobject is an isomorphism class of monomorphisms, meaning that for the above monos m, m', the morphism $f\colon S \longrightarrow S'$ is an iso.

The collection of all subobjects are denoted by $\mathrm{Sub}_{\mathscr{C}}(B)$. In a topos $\mathscr{E}$,

$$\mathrm{Sub}_{\mathscr{E}}(A) \cong \mathrm{Hom}_{\mathscr{E}}(A, \Omega) \tag{2.3}$$

holds for any object $A \in \mathsf{Ob}(\mathscr{E})$ [MM12, Sec. IV.1].

Remark 2.5 (Homoousion of Subobjects)**.** In a topos $\mathscr{E}$, the following characterisations of a subobject are equivalent [MM12, Sec. IV.1, p. 165]:

Monomorphism $m\colon S \rightarrowtail B$ As discussed above, m is a representative element of the equivalence class of monos that constitute the subobject,

Characteristic morphism $\phi\colon B \longrightarrow \Omega$ As mentioned in Definition 2.2, the characteristic morphism ϕ or ϕ_m of any mono m is such that the pullback square $\phi_m \circ m = \mathrm{true} \circ !$ commutes.

Global element $s\colon 1 \longrightarrow \mathbf{P}\,B$ It is easy to see that by exponential transposition $\phi\colon B \cong B \times 1 \longrightarrow \Omega$ corresponds to $1 \longrightarrow \Omega^B \cong \mathbf{P}\,B$. In the "set-like" interpretation, we read this as the morphism that "picks out" a "subset" S of B.

Definition 2.6 ((Epi,Mono)-Factorisation)**.** A category $\mathscr{C}$ has *(strong epi,mono)-factorisation* when every arrow $f\colon A \longrightarrow B$ factors as $f = m \circ e$, where $e\colon A \twoheadrightarrow B'$ is a strong epi and $m\colon B' \rightarrowtail B$ is a mono. We refer to the subobject represented by m as the *image* $\mathrm{Im}\, f$ of f.

Definition 2.7 (Power Object Functor)**.** A (covariant) power object functor maps each object B to $\mathbf{P}\,B$. A morphism $f\colon A \longrightarrow B$ is mapped to $\mathbf{P}\,f\colon \mathbf{P}\,A \longrightarrow \mathbf{P}\,B$, by the universal property as in Equation (2.2),

$$\begin{array}{ccc} \mathbf{P}\,A & & B \times \mathbf{P}\,A & & \\ {\scriptstyle \mathbf{P}\,f}\,\downarrow & & \downarrow{\scriptstyle \mathrm{id}_B \times \mathbf{P}\,f} & \searrow^{g} & \\ \mathbf{P}\,B & & B \times \mathbf{P}\,B & \xrightarrow[\in_B]{} & \Omega \end{array} \tag{2.4}$$

To construct g, first take any subobject of the form

$$\bullet \xrightarrow{m} A \times \mathbf{P}\, A,$$

and consider the corresponding characteristic morphism

$$A \times \mathbf{P}\, A \xrightarrow{\phi_m} \Omega.$$

From this point, we can extend subobject-monomorphism

$$\bullet \xrightarrow{m} A \times \mathbf{P}\, A \xrightarrow{f \times \mathrm{id}_{\mathbf{P}\, A}} B \times \mathbf{P}\, A,$$

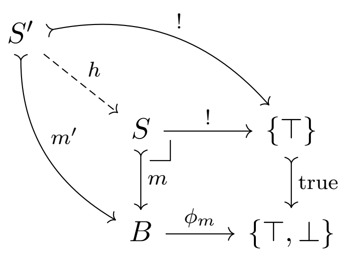
the respective characteristic morphism

$$B \times \mathbf{P}\, A \xrightarrow{\phi(f \times \mathrm{id}_{\mathbf{P}\, A}) \circ m} \Omega$$

almost takes the necessary form to serve as g. Note that this only without further issues works if f is itself mono. In general, we have to instead consider the image of $\mathrm{Im}((f \times \mathrm{id}_{\mathbf{P}\, A}) \circ m)$, and its respective characteristic morphism $\phi_{\mathrm{Im}((f \times \mathrm{id}_{\mathbf{P}\, A}) \circ m)}$. This is elaborated on in the Elephant [Joh02, A 2.3].

Example 2.8 (Examples of Elementary Toposes). What follows are some well known categories that are toposes, with some comments:

Category of sets (Sets) It is well known that **Sets** is finitely complete and has exponential objects (sets of functions). The subobject classifier is the two-element set $\Omega \cong \mathbf{2} \cong \{\top, \bot\}$. To verify that this is the subobject classifier, the following pullback square must commute:



(2.5)

where true is the intuitive injection and the characteristic morphism as mentioned in Remark 2.5 can stated directly:

$$\phi_m(b) = \begin{cases} \top & \text{if } \exists\, s \in S.\, m(s) = b \\ \bot & \text{otherwise} \end{cases}$$

While it is clear that the $! = ! \circ h$ triangle must commute for any definition of h, for $m' = m \circ h$ to commute, h must be defined as

$$h(s' \colon S') = \{\, s \colon S \mid m'(s') = m(s) \,\}\,.$$

The power-object functor in **Sets** is the power-set functor $\wp(-)$. Unsurprisingly, the power-object of any set A is the power-set $\wp(A)$.

Category of finite sets (FinSet) For finite sets, the subobject classifier remains the same, and the above constructions are likewise valid. All finite limits and exponentials exists as well. We take the finite power-set functor to be the power-object functor.

Category of Presheaves ($\mathbf{Sets}^{\mathscr{C}}$, where $\mathscr{C}$ is small) This functor category is an interesting example that doesn't derive the "set-like" structure by being immediately "set-like" to begin with. We do not concern ourselves with the precise definition here, but sketch Borceux's definition [Bor94, Ex. 5.2.5, p. 295] of the subobject classifier in $\mathbf{Sets}^{\mathscr{C}}$: The subobject classifier must be Ω a functor, defined for any object $C \in \mathsf{Ob}(\mathscr{C})$ as $\Omega(C)$ the set of all subfunctors of $\mathrm{Hom}_{\mathscr{C}}(-, C)$, and for morphisms $f\colon D \longrightarrow C$ by the pullback diagram

$$\begin{array}{ccc} \Omega(f)(S) & \longrightarrow & S \\ \downarrow & & \downarrow \\ \mathrm{Hom}_{\mathscr{C}}(-, D) & \xrightarrow{\mathrm{Hom}_{\mathscr{C}}(-, f)} & \mathrm{Hom}_{\mathscr{C}}(-, C) \end{array} \tag{2.6}$$

for all $S \in \Omega(C)$. In any case, it clearly demonstrates that the "set-like" properties of a topos can take form in ways that differ significantly from the notional equivalent of a two-element set.

(Non-example:) Category of topological spaces (Top) While the Sierpinski space over $\{0, 1\}$ with the open spaces $\{\{\}, \{1\}, \{1, 0\}\}$ would serve as a subobject classifier [nLa24, Sec. 2.2], the fact that **Top** is not cartesian closed is a sufficient condition to demonstrate that **Top** is not a topos.

Remark 2.9 (Further definitions of toposes). In Definition 2.2 a topos was defined as a finitely complete CCC with a subobject classifier. This is the canonical definition advanced by Mac Lane and Moerdijk [MM12, p. IV.1], Lambek and Scott [LS88, Def. 5.4.1, p. 339] Barr and Wells [BW00, p. 2.1], Leinster [Lei10, p. 5], Johnstone [Joh02, Def. 2.1.1], Caramello [Car18, Def. 1.3.28 (a)], Borceux [Bor94, Def. 5.1.3] and Freyd and Scedrov [FS90, p. 1.9].

Some regard this as the "category theoreticians" definition of a topos, while the "set theoreticians" defines a topos as a finitely complete category with a power object functor. From this, one can derive both the subobject classifier and all exponentials. Bell [Bel08, p. 60] gives this definition.

Historically Lawvere [Law70] and Goldblatt [Gol14, p. 4.3] presented the canonical definition with the additional requirement that the category is finitely cocomplete. Mikkelsen [Mik76, Thm. 2.3] showed how finite cocompleteness could be derived from the above canonical definition.

2.2 Internal Logic/Language of Categories

The previous section has insinuated the set-like properties that elementary toposes enjoy. What follows is a presentation of how we can exploit their structure to simplify definitions and proofs.

To motivate this point concretely: In the *internal logic of* $\mathscr{E}$, one can write down $S\colon \mathbf{P}\,A$

$$S = \{\, a\colon A \mid \phi_B(a) \wedge \phi_C(a) \,\},$$

and be confident that this describes the pullback (or intersection) of two subobjects $m\colon B \rightarrowtail A$ and $m'\colon C \rightarrowtail A$ (with the characteristic morphisms ϕ_B and ϕ_C respectively). Likewise, we can define any morphism point-wise: Raising a object to a "singleton set" by $\eta\colon A \longrightarrow \mathbf{P}\,A$ has a plain definition given by

$$\eta(a) = \{a\} = \{\, a'\colon A \mid a = a' \,\},$$

suffices, where the latter set comprehension is just a more verbose version of the first. To prove a claim like $S \cap A = S$ we can argue using extensionality of sets, that is that for some element $a\colon A$, $a \in S \cap A \iff a \in S$ must hold.

The intention here is not to prove that reasoning about categories in "the language" of sets is sound or complete. To that end, see Mac Lane's chapters on "Mitchell-Bénabou Language" and

"Kripke-Joyal Semantics" [Mac13, Sec. IV.5 and 6], Borceux's rigorous introduction of the inference rules in intuitionistic propositional [Bor94, Sec. 6.7, p. 395] and predicate calculus [Bor94, Sec. 6.8, p. 400], as well as intuitionistic set theory [Bor94, Sec. 6.9, p. 409].

It is worthwhile to contemplate the notion of "internality" in category theory for a brief moment. An instructive example is to consider the relation between $\mathrm{Hom}_{\mathscr{C}}(A,B)$ and the exponential $B^A \in \mathsf{Ob}(\mathscr{C})$. Assuming $\mathscr{C}$ is a locally small category, like **Sets**, that is to say that $\mathrm{Hom}_{\mathscr{C}}(A,B)$ is not a proper class, we can view the latter as an a "internal", as in internal to the category, representation of the latter.

A related example involves the power object $\mathbf{P}\,A$ and the collection of subobjects $\mathrm{Sub}_{\mathscr{E}}(A)$. The former gives a partial order inside the topos, while the latter is external to it. Nevertheless, the two correspond exactly to one another.

2.2.1 The Internal Language of a CCC

As a first step, recall Definition 2.1. The well-known Curry-Howard Correspondance, that associated the inhabitation of types in the simply-typed λ-calculus (STLC) and provability of propositions in minimal intuitionistic propositional logic can be extended to a categorical setting. We fix $\mathscr{C}$ to be a Closed Cartesian category throughout this subsection.

Take the tautology $(\phi \to \psi) \to (\psi \to \chi) \to \phi \to (\psi \wedge \chi)$. Figure 2.1 gives a Sequent-style natural deduction proof for the proposition. In STLC, we can inhabit the corresponding type $(A \to B) \to (B \to C) \to A \to (B \wedge C)$ by the term

$$\lambda\, f.\, \lambda\, g.\, \lambda\, a.\, (fa), (g(fa)).$$

On closer inspection, a resemblance is found between the lambda-term and the proof in Figure 2.1: The proof of a conjunction corresponds to the construction of a pair, modus ponens corresponds to λ-abstraction applications, the management of a sequent context corresponds to variable bindings.

The categorical analogue for provability or inhabitation is, in the above example, the existence of a morphism

$$1 \rightarrowtail \left(\left((B \times C)^A\right)^{B^C}\right)^{B^A} \quad \text{or by transposition} \quad B^A \times C^B \times A \longrightarrow B \times C.$$

We recognise the objects A, B, C in $\mathscr{C}$ represent atomic propositions in minimal logic and base types in STLC. Exponential objects such as B^A mirror implications and function types respectively. In general, object of $\mathscr{C}$ correspond to types or propositions, and the existence of morphisms to terms or proofs. Above we considered both a map with a terminal object and a product as the domain. These correspond to the logical judgements

$$\vdash (\phi \to \psi) \to (\psi \to \chi) \to \phi \to (\psi \wedge \chi)$$

or by the deduction theorem, whereby we "extract" the "internal" implication into "external" assumptions (and vice versa)

$$\phi \to \psi, \psi \to \chi, \phi \vdash \psi \wedge \chi.$$

Compare this to the formula $\phi \to \psi$. We know that for arbitrary propositions, the implication cannot always be satisfied. We know that a function type between two arbitrary types cannot always be inhibited by a λ-term. And likewise, a morphism $A \longrightarrow B$ must not exist between arbitrary objects (in **Sets**, $\mathrm{Hom}_{\mathbf{Sets}}(\{*\}, \{\})$ is known to be empty).

This perspective is due to Lambek [Lam86] and states that λ-calculus (or minimal propositional logic) is the "internal language" or "internal logic" of CCC. That is to say that due to the strong

$$\dfrac{\dfrac{\dfrac{\dfrac{\phi \vdash \phi}{(\phi \to \psi), (\psi \to \chi), \phi \vdash \phi}, \dfrac{(\phi \to \psi) \vdash (\phi \to \psi)}{(\phi \to \psi), (\psi \to \chi), \phi \vdash (\phi \to \psi)}}{(\phi \to \psi), (\psi \to \chi), \phi \vdash \psi}, \dfrac{\dfrac{\dfrac{\phi \vdash \phi}{(\phi \to \psi), (\psi \to \chi), \phi \vdash \phi}, \dfrac{(\phi \to \psi) \vdash (\phi \to \psi)}{(\phi \to \psi), (\psi \to \chi), \phi \vdash (\phi \to \psi)}}{(\phi \to \psi), (\psi \to \chi), \phi \vdash \psi}, \dfrac{(\psi \to \chi) \vdash (\psi \to \chi)}{(\phi \to \psi), (\psi \to \chi), \phi \vdash (\psi \to \chi)}}{(\phi \to \psi), (\psi \to \chi), \phi \vdash \chi}}{(\phi \to \psi), (\psi \to \chi), \phi \vdash (\psi \wedge \chi)}}{\vdash (\phi \to \psi) \to (\psi \to \chi) \to \phi \to (\psi \wedge \chi)}$$

Figure 2.1: A simplified Sequent-style natural deduction proof

correspondence sketched above, demonstrating the existence of a morphism is exactly equivalent to the inhabitation of a type or the proof of a formula. The appeal is that the latter two are frequently more convenient to handle than a categorical proof, and allowing ourselves to argue in these terms can simplify our proofs.

2.2.2 The Internal Language of a Topos

Mirroring the jump from Definition 2.1 (CCC) to Definition 2.2 (Toposes), the additional structure grants a more expressive "internal language". As mentioned at the beginning of Section 2.2, this approach links categorical statements to statements in intuitionistic set theory (without the axiom of choice).

Just as in Section 2.2.1, the premise is that objects corresponds to formulae/types, and morphisms to proofs. We will sketch these in the following. For more details, consult the previously mentioned literature.

Any term of type Ω is a formula in the internal language. The usual connectives $\wedge$, $\vee$, $\implies$, $\neg$ enjoy the inference rules known from intuitionistic propositional calculus.

Formulae may have free variables $x_1\colon X_1, x_1\colon X_2, \ldots, x_n\colon X_n$, that externally appear in the domain:

$$\phi\colon X_1 \times X_2 \times \cdots \times X_n \longrightarrow \Omega.$$

These can be bound by universal and existential quantifiers:

$$\exists\, x_1.\, \phi\colon X_2 \times \cdots \times X_n \longrightarrow \Omega,$$
$$\forall\, x_1.\, \phi\colon X_2 \times \cdots \times X_n \longrightarrow \Omega,$$

and are interpreted and used according the intuitionistic predicate calculus.

Recall that a subobject of A is of type Ω^A. Externally, this corresponds to an arrow $A \longrightarrow \Omega$, i.e. a formula with a free variable of type A. We can use this fact to denote a specific subobject via set-comprehension-notation:

$$\{\, a\colon A \mid \phi \,\}\,.$$

Note: This is a special case of a general definition of any morphism $f\colon A \longrightarrow B$, that we can define point-wise in the internal logic of a topos, where either

$$f(a) = \ldots \qquad \text{or} \quad a \mapsto \ldots$$

are both legal constructs that correspond to the set theoreticians expectations.

Returning to *subobjects-as-sets*, the two important applications include set membership,

$$\in_A\colon A \times \Omega^A \longrightarrow \Omega$$

that is easily recognised to just be evaluation, and extensional equality, where

$$\vdash \{\, a\colon A \mid \phi(a) \,\} = \{\, a\colon A \mid \psi(a) \,\}$$

holds, when

$$a'\colon A \vdash a' \in \{\, a\colon A \mid \phi(a) \,\} \iff a' \in \{\, a\colon A \mid \psi(a) \,\}$$

and then is simplified to

$$a'\colon A \vdash \phi(a') \iff \psi(a').$$

Example 2.10 (Power Set Functor in the Internal Logic of $\mathscr{E}$). Utilising these properties is straightforward: For example recalling Definition 2.7, an equivalent characterisation of the power object functor's morphism map in the internal logic of $\mathscr{E}$ is:

$$\mathbf{P}\,(f)\,(\mathcal{A}) := \{\, b\colon B \mid \exists\, a\colon A.\, a \in \mathcal{A} \wedge b = f(a) \,\}\,, \tag{2.7}$$

for a $f\colon A \longrightarrow B$.

2.2.3 Example: Naturality of η

As a first example of how the above notions can be employed, by proving an intuitive claim internally:

Proposition 2.11. *$\eta\colon \mathrm{Id} \Rightarrow \mathbf{P}$ is a natural transformation.*

Proof. We translate the component-wise commutative diagram

$$\begin{array}{ccc} A & \xrightarrow{\eta_A} & \mathbf{P}\,A \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle \mathbf{P}\,f} \\ B & \xrightarrow{\eta_B} & \mathbf{P}\,B \end{array}$$

directly into the internal logic of $\mathscr{E}$ as the statement

$$\vdash \mathbf{P}\,f \circ \eta_A = \eta_B \circ f.$$

As a topos is functionally extensional, we proceed to prove the equality of functions, by ensuring that the functions behave the same on all arguments,

$$a\colon A \vdash (\mathbf{P}\,f)(\eta_A(a)) = \eta_B(f(a)).$$

This in turn proposes equality of subobject. As in classical set-theory, equality is extensional, meaning subobjects are determined by their elements. Hence the above is equivalent to

$$a\colon A, b\colon B \vdash b \in (\mathbf{P}\,f)(\eta_A(a)) \iff b \in \eta_B(f(a)).$$

Recall the definition of $\eta_X(x) = \{\, x' \mid x' = x \,\} = \{x\}$. If we expand the definition of the $\mathbf{P}\,f$ as given in Equation (2.7), we have the following chain of reasoning:

$$\begin{aligned} & b \in \{\, b'\colon B \mid b' = f(a) \,\} \\ \iff & b = f(a) \\ \overset{(*)}{\iff} & \exists\, a'\colon A.\, a' = a \land b = f(a') \\ \iff & b \in \{\, b'\colon B \mid \exists\, a'\colon A.\, a' = a \land b' = f(a') \,\} \\ \iff & b \in \{\, b'\colon B \mid \exists\, a'\colon A.\, a' \in \{a\} \land b' = f(a') \,\} \\ \iff & b \in \{\, b'\colon B \mid \exists\, a'\colon A.\, a' \in \eta_A(a) \land b' = f(a') \,\} \\ \iff & b \in (\mathbf{P}\,f)(\eta_A(a)) \end{aligned}$$

As the internal logic of an arbitrary topos is intuitionistic, we can prove that the inference annotated by $(*)$ is valid using the Coq[1] proof assistant, which is founded on intuitionistic logic:

```
Parameter f : Set -> Set.
Parameter a b : Set.

Goal b = f a <-> (exists a', a' = a /\ b = f a').
Proof.
  split.
  - intro H.
    exists a.
    split.
```

[1] https://coq.inria.fr/

```
    + congruence.
    + assumption.
  - intro H.
    destruct H as [a' [H1 H2]].
    rewrite H1 in H2.
    assumption.
Qed.
```

or now `firstorder subst.` if there were a need for brevity. ■

2.2.4 Example: Extension of a Function along a Monoid

As a second example, we shall foreshadow coming developments by assuming a topos $\mathscr{E}$ with further structure:

Definition 2.12. A category $\mathscr{C}$ is countably extensive TODO

Definition 2.13. For any function $f\colon B \times A \longrightarrow \mathbf{P}\, B$, the *extension* $\overline{f(-)}\colon B \times A^\star \longrightarrow \mathbf{P}\, B$ is defined by transposing the initial algebra morphism $¡\colon A^\star \longrightarrow \mathbf{P}\,(B)^B$ of the functor $FX = 1 + A \times X$.

Proposition 2.14. *The extension of f is well behaved.*

Proof. By "well behaved", we mean to expect that

$$\overline{f(b)}(w) = \begin{cases} \{b\} & \text{if } w = \epsilon \\ \bigcup_{b' \in f(b,s)} \overline{f(b')}(w') & \text{if } sw' = w \end{cases} . \tag{2.8}$$

holds, in the internal logic of $\mathscr{E}$.

Before proceeding, we have to likewise give an internal definition of $\overline{f(-)}$ and assure ourselves that this corresponds to ¡ by making the following diagram commute

$$\begin{CD} 1 + A \times A^* @>{[\mathrm{nil},\mathrm{cons}]}>> A^* \\ @V{\mathrm{id}_1 + \mathrm{id}_A \times ¡}VV @VV{¡}V \\ 1 + A \times \mathbf{P}\,(B)^B @>{[n,c]}>> \mathbf{P}\,(B)^B \end{CD} \tag{2.9}$$

We therefore assume Equation (2.8) as the definition of ¡ or $\overline{f(b)}$, and separately define

$$\begin{aligned} n(*) &= b \mapsto \{b\}, \\ c(s,g) &= b \mapsto \bigcup_{b' \in f(b,s)} g(b'). \end{aligned}$$

By showing that Equation (2.9) commutes, we can conclude that $\overline{f(-)} = ¡$. We do this by "splitting up" the coproduct $1 + A \times A^*$ and looking at the resulting equations separately:

1. If $x = \iota_2 *$, which is to say that $w = \epsilon$, then

   $$(\overline{f(-)} \circ \mathrm{nil})(*) = ((a \mapsto b \mapsto \{b\}) \circ \mathrm{id}_1)(*)$$

   holds trivially.

2. For a non-empty word where $x = \iota_1(s, w)$ and $b\colon B$ arbitrary, consider

$$\begin{aligned}
&\overline{f(-)}(\mathrm{cons}(s,w)) \\
&= b \mapsto \bigcup_{b' \in f(b,s)} \overline{f(b')}(w) \\
&= b \mapsto \bigcup_{b' \in f(b,s)} \overline{f(-)}(w)(b') \\
&= c(s, \overline{f(-)}(w)) \\
&= (c \circ (\mathrm{id}_A \times \overline{f(-)}))(s,w)
\end{aligned}$$

■

2.3 Nondeterministic Categorical Automata

Readers not familiar with the standard construction of a nondeterministic finite automaton (NFA), should consult Hopcroft et al. [HMU06, Sec. 2.3.2, p. 57]. In the following we will be considering nondeterministic automata (NDA), generalising NFAs over arbitrary state spaces.

Recall that a NFA/NDA is represented by a tuple $(Q, \Sigma, \delta, I, F)$:

- Q a (finite) set of states,
- Σ a (finite) set representing the input alphabet,
- δ a function representing state transitions of the type $Q \times \Sigma \to Q$,
- I a subset of the set Q representing the initial states,
- F a subset of the set Q representing the accepting states.

2.3.1 Categorical NDAs

Frank et al. [FMU23, Sec. 6, p. 10] present a categorification of NDA for an arbitrary category $\mathscr{C}$ with finite limits and (epi,mono)-factorisations (c.f. Definition 2.6), again represented by a tuple $A = (Q, \Sigma, \delta, I, F)$:

- Q an object $\mathscr{C}$ of states,
- Σ a object $\mathscr{C}$ representing the input alphabet,
- a subobject $m_\delta\colon \delta \rightarrowtail Q \times \Sigma \times Q$, representing a ternary relation of legal state transitions,
- a subobject $m_I\colon I \rightarrowtail Q$, representing the initial states,
- a subobject $m_F\colon F \rightarrowtail Q$, representing the accepting states.

It is easy to see that for $\mathscr{C} = \mathbf{Sets}$ we get NDAs, and that $\mathscr{C} = \mathbf{FinSet}$ gives us NFAs. Also note that an arbitrary topos has all the necessary structure to construct a categorical automaton.

A point of clarification regarding δ, as a subobject of $Q \times \Sigma \times Q$: the first element Q is a given state, the second element the symbol in the input alphabet Σ by which the automaton transitions to the state denoted by the third element Q. In **Sets**, this relation is equivalent to functions of the types $Q \times \Sigma \longrightarrow \wp(Q)$ or $Q \longrightarrow \wp(Q)^\Sigma$.

The semantics of a particular automaton A, in the words it will accept, i.e. the "accepting runs" that start in an initial state and after processing a series of symbols $s_0, s_1, s_2, \ldots$ from the input ends up in a accepting state.

As we assume that $\mathscr{C}$ only has finite limits (let alone any coproducts), it is not countably extensive (c.f. Definition 2.12). Therefore, there is no single object that can denote all words $w = s_0, s_1, s_2, \ldots, s_n$ of arbitrary length.

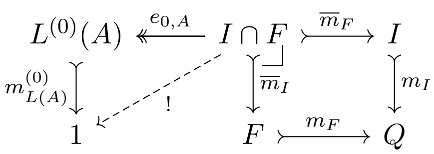


Figure 2.2: Commutative diagrams describing a NDA for $n = 0$

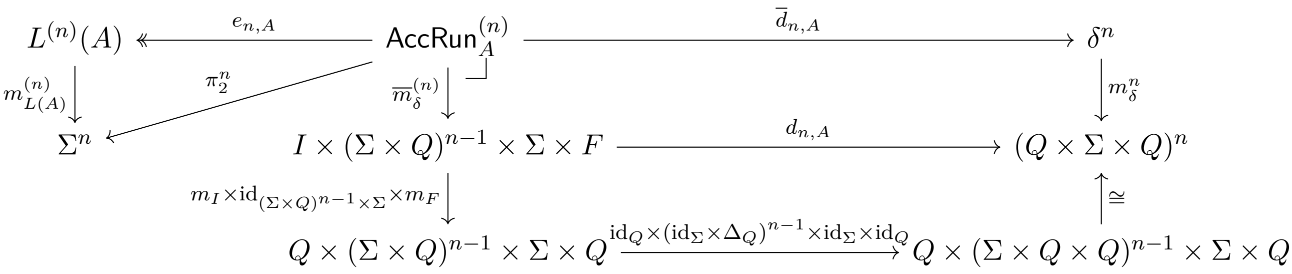


Figure 2.3: Commutative diagrams describing a NDA for $n \geq 1$

For that reason, we define a language to be a family of subobjects:

$$L \coloneqq \left(m^{(L)}_n \colon L^{(n)} \rightarrowtail \Sigma^n\right)_{n\in\mathbb{N}}, \tag{2.10}$$

where each constituent of the family is respectively defined in terms of the commutative diagrams in Figure 2.2 and Figure 2.3:

Accepted words of length $n = 0$ There is only a single accepted word of nil-length, ϵ (the empty word). An automaton accepts this iff there is an "overlap" between the initial and the accepted states. The "overlap" is expressed categorically in terms of a pullback of m_I and m_F, and denoted by $I \cap F$.

It is therefore not surprising to note that $\Sigma^0 \cong 1$ has two subobjects: $\{\} \cong 0$ and $\{1\} \cong 1 \cong \{\epsilon\}$.

If $I \cap F$ is an "empty intersection", in which case it is isomorphic to 0, then the image $\mathrm{Im}(!) = L^{(0)}(A)$ of $0 \longrightarrow 1$ is likewise 0. Otherwise, the image denotes the singleton subobject $\{\epsilon\}$. This is a satisfactory definition of $m^{(L)}_0 \colon L^{(0)} \rightarrowtail \Sigma^0$ and intuitively matches our expectations from automata theory.

Accepted words of length $n \geq 1$ For any non-empty word $w = s_0, s_1, \ldots$, we need to ensure that there is a legal "accepting run", that is to say a subobject of δ^n, where the right state of the i'th entry matches the left state of the $i+1$'th entry. In Figure 2.3 we describe the accepting runs by a pullback of the subobject $m^n_\delta$of n arbitrary legal transitions and the morphism $d_{n,A}$ that reorders a

$$q_0, s_0, q_1, s_1, s_2, \ldots, q_{n-2}, s_{n-2}, q_{n-1}, s_{n-1}, q_n$$

where $q_0 \in I \subseteq Q$ and $q_n \in F \subseteq Q$, by duplicating each mid-state

$$q_0, s_0, q_1, q_1, s_1, q_2, \ldots, q_{n-2}, s_{n-2}, q_{n-1}, q_{n-1}, s_{n-1}, q_n$$

and then utilising the associativity of products to re-order the product into the intended form:

$$(q_0, s_0, q_1), (q_1, s_1, q_2), \ldots, (q_{n-2}, s_{n-2}, q_{n-1}), (q_{n-1}, s_{n-1}, q_n).$$

The accepted words are of course a subobject of $m^{(n)}_{L(A)} \colon L^{(n)}(A) \rightarrowtail \Sigma^n$, that correspond to the image or equivalently monomorphism of the (epi,mono)-factorisation of the morphism $\pi^n_2 \colon \mathsf{AccRun}^{(n)}_A \longrightarrow \Sigma^n$ projecting the symbols in the input alphabet that constitute the accepting run.

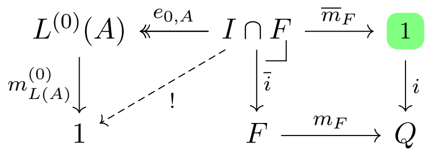


Figure 2.4: Commutative diagrams describing a SISNDA for $n = 0$. The main difference to Figure 2.2 is emphasised in green.

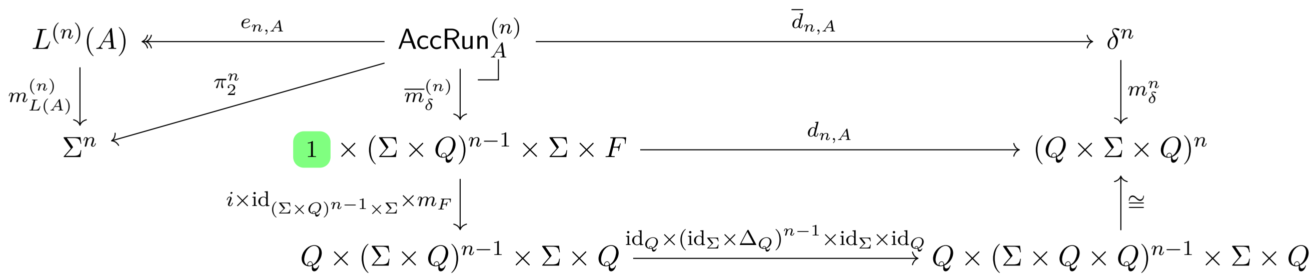


Figure 2.5: Commutative diagrams describing a SISNDA for $n \geq 1$. The main difference to Figure 2.3 is emphasised in green.

2.3.2 Categorical Automata with a Single Initial State

It will present itself as useful to consider automata with a single initial state. We would like to assure ourselves that this matter of convenience does not restrict the generality of our results.

The definition of a single-initial-state NDA (SISNDA) is mostly equivalent to that of a NDA, with the difference that we do not consider a subobject I of Q to denote the possible initial states, but just an object Q (or equivalently a global element $i\colon 1 \rightarrowtail Q$).

It should not be surprising, that the subobjects of L, that replace m_I with i are mostly similar. These are given in Figure 2.4 and Figure 2.5. Their interpretation is also equivalent to the diagrams Figure 2.2 and Figure 2.3.

Theorem 2.15 (Single-State Automata)**.** *For any NDA, there exists exactly one SISNDA with the same semantics.*

Proof. Admitted. ■

2.4 Eilenberg-Moore Algebras and their Semantics

For readers unfamiliar with F-Coalgebras (sometimes also referred to as "algebra over an endofunctor"), please consult Jacobs [Jac17].

Recall that a general F-Coalgebra is specified by an endofunctor F over a category $\mathscr{C}$. In the following we will consider special cases of F, that are defined by composing an arbitrary functor G with a monad functor T. We follow and recapitulate the results of Jacobs et al. [JSS12] — omitting proofs — to demonstrate how and under which conditions this ensures the existence of a terminal coalgebra.

As a running example we will consider non-deterministic automata in **Sets**, conventionally given by the endofunctor $FQ = 2 \times \wp(Q)^\Sigma$. F is equivalently defined as the composition of $GQ = 2 \times Q^\Sigma$ and $TQ = \wp(Q)$ (by the well-known fact that powersets form a monad).

Recall that any monad

$$(T\colon \mathscr{C} \longrightarrow \mathscr{C}, \mu\colon TT \Rightarrow T, \eta\colon \mathrm{Id} \Rightarrow X)$$

satisfies the following requirements:

$$\begin{array}{ccc} TX & \xrightarrow{\eta_{TX}} & TTX \\ {\scriptstyle T(\eta_X)}\downarrow & \diagdown\!\!\diagdown & \downarrow{\scriptstyle \mu_X} \\ TTX & \xrightarrow{\mu_X} & TX \end{array} \qquad \begin{array}{ccc} TTTX & \xrightarrow{\mu_{TX}} & TTX \\ {\scriptstyle T(\mu_X)}\downarrow & & \downarrow{\scriptstyle \mu_X} \\ TTX & \xrightarrow{\mu_X} & TX \end{array} \tag{2.11}$$

Definition 2.16 (Eilenberg-Moore Algebra)**.** A *Eilenberg-Moore algebra* over a monad (T, μ, η) is a morphism $\alpha\colon TX \longrightarrow X$ such that

$$\begin{array}{ccc} X & \xrightarrow{\eta_X} & T(X) \\ & \diagdown\!\!\diagdown & \downarrow{\scriptstyle a} \\ & & X \end{array} \qquad \begin{array}{ccc} TTX & \xrightarrow{\mu_X} & TX \\ {\scriptstyle T(\alpha)}\downarrow & & \downarrow{\scriptstyle \alpha} \\ TX & \xrightarrow{\alpha} & T \end{array} \tag{2.12}$$

both commute.

Definition 2.17 (Eilenberg-Moore Category)**.** The *category of Eilenberg-Moore Algebras* $\mathcal{EM}(T)$ is a restriction of $\mathbf{Alg}(T)$ to functors that satisfy Equation (2.12). Notions such as initial algebras in $\mathcal{EM}(T)$ exist analogously as in $\mathbf{Alg}(T)$.

Definition 2.18 ($\mathcal{EM}$-law)**.** For a monad $(T\colon \mathscr{C} \longrightarrow \mathscr{C}, \mu, \eta)$ and an arbitrary endo-functor $G\colon \mathscr{C} \longrightarrow \mathscr{C}$, a *distributive* $\mathcal{EM}$*-law* is a natural transformation

$$\rho\colon TG \Rightarrow GT, \tag{2.13}$$

such that the following two commutative diagrams commute:

$$\begin{array}{ccccc} & & GX & & \\ & {\scriptstyle \eta_{GX}}\swarrow & & \searrow{\scriptstyle G(\eta_X)} & \\ TGX & & \xrightarrow{\quad\rho_X\quad} & & GTX \end{array} \qquad \begin{array}{ccccc} TTGX & \xrightarrow{T(\rho_X)} & TGTX & \xrightarrow{\rho_{TX}} & GTTX \\ \downarrow{\scriptstyle \mu_{GX}} & & & & \downarrow{\scriptstyle G(\mu_X)} \\ TGX & & \xrightarrow{\quad\rho_X\quad} & & GTX \end{array} \tag{2.14}$$

Remark 2.19 (Interpretation in **Sets** for Automata)*.* We take $TQ = \wp(Q)$ and $GQ = 2 \times Q^\Sigma$. Instantiating the type of $\mathcal{EM}$-law pointwise

$$\rho_Q\colon \wp\left(2 \times Q^\Sigma\right) \Rightarrow 2 \times \wp(Q)^\Sigma, \tag{2.15}$$

one can see, that this is describing the transformation of a set of deterministic automata, into a single-non-deterministic automaton, which is supposed to be well-behaved in the intuitive sense by Equation (2.14).

Following Jacobs et al. [JSS12, Sec. 5.1, p. 117] as define the $\mathcal{EM}$-law as $\rho_Q = \langle \rho_{Q,1}, \rho_{Q,2} \rangle$, where $\rho_{Q,1}\colon \wp\left(2 \times Q^\Sigma\right) \longrightarrow 2$ is defined as

$$\rho_{Q,1}(A) = \exists\, a \in A.\, \pi_1(a) \tag{2.16}$$

that determines if any of the automata in the powerset is in an accepting state, and the second component of the type $\rho_{Q,2}\colon \wp\left(2 \times Q^\Sigma\right) \longrightarrow \wp(Q)^\Sigma$ as

$$\rho_{Q,2}(A) = s \mapsto \{\, q\colon Q \mid \exists\, a \in A.\, q = \pi_2(a)(s) \,\} \tag{2.17}$$

that applies the transition function to each automaton, aggregating the resulting deterministic states as the next non-deterministic state.

We omit the proof that this definition satisfies the $\mathcal{EM}$-law, as that result will follow from `TODO` by verifying the claim in an arbitrary topos.

Definition 2.20 (Liftings of Functors to $\mathcal{EM}$-categories)**.** Any functor $G\colon \mathscr{C} \longrightarrow \mathscr{C}$ can be *lifted* from $\mathscr{C}$ to an Eilenberg-Moore category as $\hat{G}\colon \mathcal{EM}(T) \longrightarrow \mathcal{EM}(T)$, given a $\mathcal{EM}$-law (c.f. Definition 2.18) $\rho\colon TG \Rightarrow GT$.

The object-map of $\hat{G}$ is defined as

$$\left(TX \xrightarrow{a} X\right) \mapsto \left(TGX \xrightarrow{\rho_X} GTX \xrightarrow{G(a)} GX\right) \tag{2.18}$$

and the morphism map of $\hat{G}$ as

$$f \mapsto G(f). \tag{2.19}$$

Jacob's construction relies on the existence of a final (Z, ζ) in $\mathbf{Coalg}(G)$. Given ζ, we can construct another object in $\mathbf{Coalg}(G)$

$$TZ \xrightarrow{T(\zeta)} TGZ \xrightarrow{\rho_Z} GTZ,$$

and know that there must be a unique morphism $\alpha\colon TZ \longrightarrow Z$ from the latter to the former:

$$\begin{array}{ccc} TZ & \xrightarrow{T(\zeta)\circ\rho} & GTZ \\ {\scriptstyle\alpha}\downarrow & & \downarrow{\scriptstyle G(\alpha)} \\ Z & \xrightarrow{\zeta} & GZ \end{array} \qquad \text{in } \mathscr{C}$$

Recognising that the morphism β constitutes an Eilenberg-Moore Algebra, a change of perspective reveals an object in $\mathbf{Coalg}(\hat{G})$:

$$\begin{matrix} TZ \\ \downarrow\alpha \\ Z \end{matrix} \xrightarrow{\zeta} \hat{G}\begin{pmatrix} TZ \\ \downarrow\alpha \\ Z \end{pmatrix} \qquad \text{in } \mathcal{EM}(\mathrm{T})$$

as $\hat{G}(\alpha) = TGZ \xrightarrow{G(\alpha)\circ\rho} GZ$.

Example 2.21 (Semantic Map). In our example for $GQ = 2 \times Q^{\Sigma}$, the carrier of the final coalgebra is $Z = \wp(\Sigma^\star)$. Given a $\mathcal{EM}$-law that distributes $\wp(-)$ over G and the finality of α, the construction provides a map from an arbitrary state Q to the set of accepted words,

$$[\![-]\!] : Q \xrightarrow{\eta_Q} \mathbf{P}\,Q \xrightarrow{\alpha} \wp(\Sigma^\star)\,.$$

In Section 3.1 we will re-use these results from **Sets**, and verify that the construction is legal in an arbitrary topos $\mathscr{E}$.

2.5 Graded Monads

The second semantic approach will revolve around *graded monads*, as presented by Milius et al. [MPS15]. All of the following definitions are taken from that reference.

Definition 2.22 (Graded Monad)**.** A *graded monad* on $\mathscr{C}$ is a family of endofunctors

$$(M_n\colon \mathscr{C} \longrightarrow \mathscr{C})_{n\in\mathbb{N}},$$

a natural transformation $\eta\colon \mathrm{Id} \Rightarrow M_0$ (*unit*) and a family of natural transformations (*multiplication*)

$$\left(\mu^{n,k}\colon M_n M_k \Rightarrow M_{n+k}\right)_{n\in\mathbb{N},m,\mathbb{N}}.$$

These satisfy the *unit* and *associativity laws*:

$$\begin{array}{ccc} M_n & \xrightarrow{\eta M_n} & M_0 M_n \\ {\scriptstyle M_n\eta}\downarrow & \diagdown\!\!\diagdown & \downarrow{\scriptstyle \mu^{0,n}} \\ M_n M_0 & \xrightarrow[\mu^{n,0}]{} & M_n \end{array} \qquad \begin{array}{ccc} M_n M_k M_m & \xrightarrow{M_n\mu^{k,m}} & M_n M_{k+m} \\ {\scriptstyle \mu^{n,k}M_m}\downarrow & & \downarrow{\scriptstyle \mu^{n,k+m}} \\ M_{n+k} M_m & \xrightarrow[\mu^{n+k,m}]{} & M_{n+k+m} \end{array}$$

Definition 2.23 (Graded Trace Semantics)**.** For a F-Coalgebra X, γ the *graded trace semantics* consist of

- a graded monad $(M_n)_{n\in\mathbb{N}}$,
- a natural transformation $\alpha\colon G \Rightarrow M_1$.

Notation 2.24 ((Graded) Kleisli Star $(-)^*_n$)**.** For a $f\colon X \longrightarrow M_k Y$, we write

$$f^*_n = \mu^{n,k}_Y \circ M_n f\colon M_n X \longrightarrow M_{n+k} Y. \tag{2.20}$$

Definition 2.25 (α-pretrace sequence)**.** For a graded trace semantics $\big((M_n)_{n\in\mathbb{N}}, \alpha\big)$, the *$\alpha$-pretrace sequence* is a family of maps

$$\Big(\gamma^{(n)}\colon X \longrightarrow M_n 1\Big)_{n\in\mathbb{N}}$$

defined by

$$\begin{aligned} \gamma^{(0)} &\coloneqq \eta_X\colon X \longrightarrow M_0 1 \\ \gamma^{(n+1)} &\coloneqq (\gamma^{(n)})^*_1 \circ \alpha_X \circ \gamma\colon X \longrightarrow M_{n+1} 1 \end{aligned}$$

Notation 2.26 (up-to-n-times product)**.** For the sake of legibility, in the following we will abbreviate:

$$\Sigma^{<n} \coloneqq \coprod_{i=0}^{n} \Sigma^i,$$

where $n \in \mathbb{N}$.

Example 2.27 (NDA in **Sets**). For a F-Coalgebra $FQ = 2 \times \wp(Q)^\Sigma$, we can choose

$$\Big(M_n Q = \wp\left(\Sigma^{<n-1} + Q \times \Sigma^n\right)\Big)_{n\in\mathbb{N}}$$

and a $\alpha\colon G \Rightarrow M_1$ or equivalently $\alpha\colon 2 \times \wp(-)^\Sigma \Rightarrow \wp\left(\Sigma^{<0} + - \times \Sigma\right)$ defined as

$$\alpha((o,t)) = \{\, \iota_1 1 \mid o \,\} \cup \{\, \iota_2(\sigma, x) \mid t(\sigma) = x \,\}. \tag{2.21}$$

It is easy to see that M_n forms a graded monad with the multiplication

$$\mu^{n,m}_Q\colon \wp\left(\left(\Sigma^{<n-1}\Sigma^i\right) + \Sigma^n \times \wp\left(\left(\Sigma^{<m-1}\Sigma^i\right) + \Sigma^m \times Q\right)\right) \longrightarrow \wp\left(\left(\Sigma^{<n+m-1}\Sigma^i\right) + \Sigma^{n+m} \times Q\right)$$

defined as

$$\begin{aligned} \mu^{n,m}_Q(S) \coloneqq\ &\{\, \iota_2(wv, V) \mid \iota_2(w, W) \in S, \iota_2(v, V) \in W \,\} && (2.22)\\ &\cup \{\, \iota_1(wv) \mid \iota_2(w, W) \in S, \iota_1(v) \in W \,\} && (2.23)\\ &\cup \{\, \iota_1(w) \mid \iota_1(w) \in S \,\}, && (2.24) \end{aligned}$$

and unit being $\eta_Q(q) \coloneqq \{\iota_1((q,\epsilon))\}$, as this is just a variant of the standard power-set monad in **Sets**.

The graded trace semantics $\gamma^{(n)}\colon Q \longrightarrow M_n 1$ for a $q\colon Q$ in this case are

$$\gamma^{(0)}(q\colon Q) = \eta_Q(q) = \{\iota_2((q,\epsilon))\}\colon M_0 1 \tag{2.25}$$

for the base-case $n = 0$, and for $n+1$

$$\begin{aligned}
\gamma^{(n+1)}(q) &= \left(\left(\gamma^{(n)}\right)^*_1 \circ \alpha_{FQ} \circ \gamma\right)(q) \\
&= \left(\mu_1^{1,n} \circ M_1\gamma^{(n)} \circ \alpha_{FQ} \circ \gamma\right)(q) && \text{By Equation (2.20)} \\
&= \left(\mu_1^{1,n} \circ \wp\left(\mathrm{id}_1 + \mathrm{id}_\Sigma \times \gamma^{(n)}\right) \circ \alpha_{FQ} \circ \gamma\right)(q) \\
&= \{\, \iota_1\, \epsilon \mid \iota_1\, 1 \in S \,\} \cup \{\, \iota_1(\sigma\vec{\sigma}) \mid \iota_2(\sigma, \iota_2\, \vec{\sigma}) \in S \,\} \\
&\cup \{\, \iota_2(\sigma\vec{\sigma}, x) \mid \iota_2(\sigma, \iota_1(\vec{\sigma}, x)) \in S \,\} \\
&= \{\, \iota_1\, \epsilon \mid \pi_1(\gamma(q)) \,\} && \text{(a)} \\
&\cup \left\{\, \iota_1(\sigma\vec{\sigma}) \,\middle|\, \exists\, q' \in \gamma(q)(\sigma).\, \iota_1\, \vec{\sigma} \in \gamma^{(n)}(q') \,\right\} && \text{(b)} \\
&\cup \left\{\, \iota_2(\sigma\vec{\sigma}, x) \,\middle|\, \exists\, q' \in \gamma(q)(\sigma).\, \iota_2(\vec{\sigma}, x) \in \gamma^{(n)}(q') \,\right\} && \text{(c)}
\end{aligned}$$

We understand that ι_1-injections represent the accumulated accepted words. From a given state q, these are either (a) the empty word if the current state is already accepting, or in (b) are accepting after a single-step transition . In (c) we accumulate the transitions of length n and the respective states that these words correspond to as ι_2-injections

By throwing out ι_2-injections, we arrive at the accepted words up to a length of n:

$$\left(\pi_{(n+1)} \circ \gamma^{(n+1)}\right)(q) = \left\{\, \vec{\sigma} \,\middle|\, \iota_1\, \vec{\sigma} \in \gamma^{(n+1)}(q) \,\right\}. \tag{2.26}$$

We will review these results and their applications in the context of an arbitrary topos $\mathscr{E}$ in Chapter 4.

3 Categorical Automata and Coalgebras

3.1 Topos Semantics of a Coalgebra

To investigate the relation of a coalgebra and a categorical automaton, we will begin by describing the semantics of an coalgebra in an arbitrary topos. The functor of the coalgebra we will be considering shall be

$$GQ = \Omega \times \mathbf{P}(Q)^{\Sigma}. \tag{3.1}$$

We shall proceed by the approach sketched in Section 2.4, regarding Equation (3.1) as the composition FT of the functors

$$FQ = \Omega \times Q^{\Sigma}$$
$$TQ = \mathbf{P}\, Q$$

where T has a monadic structure (T, μ, η).

Before attempting to construct the semantic map $[\![-]\!] : Q \longrightarrow \wp(\Sigma^{\star})$, we have to verify that the necessary prerequisites hold.

3.1.1 Validity of the $\mathcal{EM}$-law in a Topos

In this section we will consider the $\mathcal{EM}$-law (c.f. Definition 2.18), i.e. the natural transformation

$$\rho\colon\ \mathbf{P}\left(\Omega \times -^{\Sigma}\right) \Rightarrow \Omega \times \mathbf{P}(-)^{\Sigma} \tag{3.2}$$

and ensure that it satisfies the conditions listed in Equation (2.14). In the following, we will reuse the **Sets**-definition of ρ from Equation (2.16), as it is just as valid in any topos $\mathscr{E}$.

Lemma 3.1. *ρ is a natural transformation.*

Proof. For the sake of an overview, we want to ensure that the following diagram commutes:

$$\begin{array}{ccccc} Q & & \mathbf{P}\left(\Omega \times Q^{\Sigma}\right) & \xrightarrow{\rho_Q} & \Omega \times \mathbf{P}(Q)^{\Sigma} \\ \downarrow f & & \downarrow T(F(f)) & & \downarrow F(T(f)) \\ Q' & & \mathbf{P}\left(\Omega \times Q'^{\Sigma}\right) & \xrightarrow{\rho'_Q} & \Omega \times \mathbf{P}(Q')^{\Sigma} \end{array}$$

We can rephrase this internally as the statement

$$a\colon\ \mathbf{P}\left(\Omega \times Q^{\Sigma}\right), f\colon Q \longrightarrow Q' \vdash \rho_{Q'}(TF(f)(a)) = FT(f)(\rho_Q(a)),$$

that we prove by a chain of equations for each of the two components:

1. On the left component (on which f has no effect), consider

$$\begin{aligned} & \pi_1(\rho_{Q'}(TF(f)(a))) \\ \Longleftrightarrow\ & \rho_{Q',1}(TF(f)(a)) \\ \Longleftrightarrow\ & \rho_{Q,1}(a) \iff \qquad \pi_1\rho_Q(a) \iff \pi_1 FT(f)(\rho_Q(a)) \end{aligned}$$

2. For the right component, consider

$$
\begin{aligned}
&\pi_2(\rho_{Q'}(TF(f)(a))) \\
=&\rho_{Q',2}(TF(f)(a)) \\
=&\{\, q \colon Q' \mid ???\,\}\, \rho_{Q',2}(TF(f)(a)) \\
=&\pi_2 FT(f)(\rho_Q(a))
\end{aligned}
$$

■

3.2 Topos Semantics of a Categorical Automaton

$$I \cap F = \{\, q \in Q \mid (\operatorname{char} I)(q) \wedge (\operatorname{char} F)(q) \,\} \tag{3.3}$$

$$R_{n,A} = \left\{ a \in \delta^n \;\middle|\; \underbrace{\pi_1(\pi_1(a)) \in I}_{\substack{\text{begins in an}\\\text{initial state}}} \wedge \underbrace{\pi_3(\pi_n(a)) \in F}_{\text{ends in final state}} \wedge \underbrace{\forall\, 1 \le i < n.\, \pi_3(\pi_i(a)) = \pi_1(\pi_{i+1}(a))}_{\text{all transitions are legal}} \right\}. \tag{3.4}$$

$$\overline{d}_{n,A}(a) = a \tag{3.5}$$

$$\overline{m}^n_\delta(a) = \left\langle \pi_1 \circ \pi_1, \langle \pi_2, \pi_3 \rangle^{n-1}, \pi_2 \circ \pi_n, \pi_3 \circ \pi_n \right\rangle (a). \tag{3.6}$$

$$\vdash d_{n,A} \circ \overline{m}^n_\delta = m^n_\delta \circ \overline{d}_{n,A} = \iota_{(Q \times \Sigma \times Q)^n}, \tag{3.7}$$

$$\operatorname{Pb}(d_{n,A}, m^n_\delta) = \{\, r \in (Q \times \Sigma \times Q)^n \mid r \in \operatorname{Im}(d_{n,A}) \wedge r \in \operatorname{Im}(m^n_\delta) \,\} \tag{3.8}$$

$$
\begin{aligned}
&\pi_{1,1}(r) \in I \wedge \pi_{3,n}(r) \in F \wedge \forall\, 1 \le i < n.\, \pi_{3,i}(r) = \pi_{1,i+1}(r) \\
\Longleftrightarrow\ &\exists\, a \colon I \times (\Sigma \times Q)^{n-1} \times \Sigma \times F.\, d_{n,A}(a) = r,
\end{aligned} \tag{3.9}
$$

$$p_{n,A} \circ \overline{m}^{(n)}_\delta, \text{ or } \pi^n_2 \circ m^n_\delta \circ \overline{d}_{n,A}, \tag{3.10}$$

$$\pi_{n,A}(a) = \pi^n_2(a) \tag{3.11}$$

$$\operatorname{Im}(\pi_{n,A}) = \left\{ \vec{\sigma} \in \Sigma^n \;\middle|\; \exists\, a \in \mathsf{AccRun}^{(n)}_A.\, \pi_{n,A}(a) = \vec{\sigma} \right\} \tag{3.12}$$

$$\{\, \vec{\sigma} \in \Sigma^n \mid \exists\, a \in \delta^n.\, \pi_{1,1}(a) \in I \wedge \pi_{3,n}(a) \in F \wedge (\forall\, i < n.\, \pi_{3,i}(a) = \pi_{1,i+1}(a)) \implies \pi^n_2(a) = \vec{\sigma} \,\}. \tag{3.13}$$

3.3 Equivalence of Descriptions

The remainder of this chapter indents to equate coalgebras and categorical automata. In other words, do the semantics of both systems (as respectively given in Equation (3.12) and **??**) coincide?

It is easy to see that it is not possible to naively compare the two semantics:

$$\left\{ \vec{\sigma} \in \Sigma^n \;\middle|\; \exists\, a \in \mathsf{AccRun}_A^{(n)}.\, \pi_{n,A}(a) = \vec{\sigma} \right\} = \left\{ \vec{\sigma} \in \Sigma^\star \;\middle|\; o(\overline{t(q)}(\vec{\sigma})) = \text{true} \right\},$$

as the left hand side only describes words of some specific length n, while the right hand side denotes all words.

We proceed by restricting $[\![-]\!]$ to words of a fixed length:

$$\begin{aligned} [\![q]\!]_n &\coloneqq \left\{ \vec{\sigma} \in \Sigma^\star \;\middle|\; \vec{\sigma} \in \left\{ \vec{\sigma} \in \Sigma^\star \;\middle|\; o(\overline{t(q)}(\vec{\sigma})) = \text{true} \right\} \wedge \|\vec{\sigma}\| = n \right\} \\ &= \left\{ \vec{\sigma} \in \Sigma^\star \;\middle|\; o(\overline{t(q)}(\vec{\sigma})) = \text{true} \wedge \|\vec{\sigma}\| = n \right\} \\ &= \left\{ \vec{\sigma} \in \Sigma^n \;\middle|\; o(\overline{t(q)}(\vec{\sigma})) = \text{true} \right\} \\ &= \{ \sigma_1 \ldots \sigma_n \in \Sigma^n \mid o(t(\cdots t(q)(\sigma_1)\cdots))(\sigma_n) = \text{true} \} \end{aligned} \tag{3.14}$$

The other apparent issue is that the above definitions do not speak of an automaton in the same terms: The left hand side relies on $\mathsf{AccRun}_A^{(n)}$ and the right hand side uses on o and t.

Remark 3.2. Any categorical automaton $A = (Q, \Sigma, \delta, I, F)$ has an equivalent coalgebraic representation $Q \xrightarrow{\langle o,t \rangle} \Omega \times \mathbf{P}\,(Q)^\Sigma$ with initial states I. In the internal logic of $\mathscr{E}$, we can define $o\colon Q \longrightarrow \Omega$ and $t\colon Q \longrightarrow \mathbf{P}\,(Q)^\Sigma$,

$$o(q) = (q \in F) \tag{3.15}$$

$$t(q) = \sigma \mapsto \{ q' \mid (q, \sigma, q') \in \delta \} \tag{3.16}$$

On these grounds, we can consider a modified semantic map for coalgebras, with the notable difference that we are considering the semantics of potentially multiple initial states:

$$[\![I]\!]_n^* \coloneqq \left\{ \vec{\sigma} \in \Sigma^n \;\middle|\; \exists\, q \in I.\, o(\overline{t(q)}(\vec{\sigma})) = \text{true} \right\} \tag{3.17}$$

Theorem 3.3. *The accepted-runs semantics of an automaton in a topos $\mathscr{E}$ coincides with the language semantics of the corresponding coalgebra.*

In other words, given an categorical automaton A and the a coalgebra as described in Remark 3.2, the $[\![I]\!]_n^*$ and the image of $\pi_{n,A}(A)$ coincide for $n > 0$:

$$\left\{ \vec{\sigma} \in \Sigma^n \;\middle|\; \exists\, q \in I.\, o(\overline{t(q)}(\vec{\sigma})) = \text{true} \right\} = \left\{ \vec{\sigma} \in \Sigma^n \;\middle|\; \exists\, a \in \mathsf{AccRun}_A^{(n)}.\, \pi_{n,A}(a) = \vec{\sigma} \right\} \tag{3.18}$$

For $n = 0$, we instead have

$$\exists\, q \in I.\, o(q) \iff I \cap F \ncong \varnothing. \tag{3.19}$$

Proof. For the empty word, i.e. $n = 0$, it is easy to see that

$$\begin{aligned} &\exists\, q \in I.\, o(q) \iff && \exists\, q \in I.\, q \in F \\ \iff &\exists\, q \in Q.\, q \in I \wedge q \in F && \text{(by } I \subseteq Q) \\ \iff &\{ q \in Q \mid q \in I \wedge q \in F \} \ncong \varnothing \\ \iff &I \cap F \ncong \varnothing \end{aligned}$$

holds.

For $n > 0$, and given a $\vec{\sigma} \in \Sigma^n$ we can prove Equation (3.18) by extensionality. Note that we will argue the legality of the step marked by $(*)$ below:

$$
\begin{aligned}
& \vec{\sigma} \in [\![I]\!]^*_n \\
\iff & \vec{\sigma} \in \left\{ \vec{\sigma} \in \Sigma^n \;\middle|\; \exists\, q \in I.\, o(\overline{t(q)}(\vec{\sigma})) = \text{true} \right\} \\
\iff & \exists\, q \in I.\, o(\overline{t(q)}(\vec{\sigma})) = \text{true} \\
\iff & \exists\, q \in I.\, \exists\, q_1 \in t(q)(\sigma_1). \cdots \exists\, q_n \in t(q_{n-1})(\sigma_n).\, o(q_n) = \text{true} \\
\iff & \exists\, q \in I.\, \exists\, q_1 \in \{\, \tilde{q} \mid (q, \sigma_1, \tilde{q}) \in \delta \,\}. \cdots \exists\, q_n \in \{\, \tilde{q} \mid (q_{n-1}, \sigma_n, \tilde{q}) \in \delta \,\}.\, o(q_n) = \text{true} \\
\iff & \exists\, q \in I.\, \exists\, (q, \sigma_1, q_1) \in \delta. \cdots \exists\, (q_{n-1}, \sigma_n, q_n) \in \delta.\, o(q_n) = \text{true} \\
\iff & \exists\, q \in I.\, \exists\, (q, \sigma_1, q_1) \in \delta. \cdots \exists\, (q_{n-1}, \sigma_n, q_n) \in \delta.\, q_n \in F \\
\overset{(*)}{\iff} & \exists\, a \in \delta^n.\, (\pi_{1,1}(a) \in I \wedge \pi_{3,n}(a) \in F \wedge (\forall\, i < n.\, \pi_{3,i}(a) = \pi_{1,i+1}(a)) \implies \pi_{n,A}(a) = \vec{\sigma}) \\
\iff & \exists\, a \in \mathsf{AccRun}^{(n)}_A.\, \pi_{n,A}(a) = \vec{\sigma} \\
\iff & \vec{\sigma} \in \left\{ \vec{\sigma} \in \Sigma^n \;\middle|\; \exists\, a \in \mathsf{AccRun}^{(n)}_A.\, \pi_{n,A}(a) = \vec{\sigma} \right\} \\
\iff & \vec{\sigma} \in L^{(n)}(A)
\end{aligned}
$$

Considering both directions of $(*)$ separately,

The $\implies$ direction We begin with a word $\vec{\sigma}$ accepted by the coalgebra, and intend to demonstrate that it is part of the accepted run semantics. To this end we have to construct a satisfactory δ^n.

Note that the

$$(q, \sigma_1, q_1), (q_1, \sigma_2, q_2), \ldots (q_{n-2}, \sigma_{n-1}, q_{n-1}), (q_{n-1}, \sigma_n, q_n)$$

form a tuple of the type δ^n. Refer to this tuples a.

By assumption $\pi_{1,1}(a) = q \in I$, $\pi_{n,3}(a) = q_n \in F$. Furthermore, each pair of consequitive triples (q_{i-1}, σ_i, q_i) and $(q_i, \sigma_{i+1}, q_{i+1})$ in a coincide in their third and first element.

By applying $\pi_{n,A}$ (defined as π_2^n) to a, we receive $\vec{\sigma}$ by construction.

Therefore, a is a fit candidate for the witness, as it satisfies both the antecedent and the consequent.

The $\impliedby$ direction In this case we assume a word accepted by the categorical automaton, and indent to demonstrate that it is also accepted by the coalgebra.

Taking $\pi_{1,1}(a)$ as a witness q, we know that $o(\pi_{3,n}(a)) = \text{true}$ and that for every a_i $(1 \leq 1 < n)$, there exist elements in δ where $\pi_3(a_i) = \pi_1(a_{i+1})$.

Therefore, we know that by iterating over $\vec{\sigma}$, in each case $t(q_i)(\sigma_i)$ will include a $(q_i, \sigma_i, q_{i+1}) \in \delta$, such that a $t(q_{i+1})(\sigma_{i+1})$ will include a $(q_{i+1}, \sigma_{i+1}, q_{i+2}) \in \delta$, and so forth. ■

4 Categorical Automata and Graded Monads

The results in Chapter 3 require $\mathscr{E}$ to not only be a topos, but be countably extensive. This was the case, as the Eilenberg-Moore Semantics of a Coalgebra map to $\Sigma^\star = \coprod_{i<\omega} \Sigma^i$, which is not a finite limit and hence not a general construction available to us in an arbitrary topos (as for example **FinSet**).

Section 2.5 presented an alternative semantic approach involving graded monads. Crucially, graded semantics involve a family of maps with "depth limited" codomains.

Conveniently it is possible to restate all the definitions from Example 2.27 (defined for **Sets**) in the internal logic of $\mathscr{E}$, as these do not rely on non-intuitionistic constructions. Specifically recall Equation (2.26) that maps a state to the set of accepted words up to a determined length.

4.1 Verifying the Graded-Monad Laws

It is first necessary to assure ourselves, that M_n constitutes a graded monad, as defined in Definition 2.22.

To recapitulate, the family of endofunctors is given by

$$\left(M_n \coloneqq \mathbf{P}\left(\Sigma^{<n} + \Sigma^n \times -\right) : \mathscr{E} \longrightarrow \mathscr{E}\right)_{n\in\mathbb{N}}. \tag{4.1}$$

From this, can infer the definition of the unit natural transpformation

$$\eta_X(q) \coloneqq \mathbf{P}(\underbrace{\Sigma^{<0}}_{0} + \underbrace{\Sigma^0}_{1} \times -)(q) = \{\iota_1(\epsilon, q)\}, \tag{4.2}$$

here given pointwise, and of multiplication

$$\begin{aligned} \mu_Q^{n,m}(S) \coloneqq\ & \{\, \iota_1(wv, V) \mid \iota_1(w, W) \in S, \iota_1(v, V) \in W \,\} \\ \cup\ & \{\, \iota_2(wv) \mid \iota_1(w, W) \in S, \iota_2(v) \in W \,\} \\ \cup\ & \{\, \iota_2(w) \mid \iota_2(w) \in S \,\}, \end{aligned} \tag{4.3}$$

equivalently to the **Sets**-definition from Equation (2.22).

Proposition 4.1. *The above M_n satisfies the* graded unit law*.*

Proof. The equational statement is $\mu_X^{0,n} \circ \eta_X(M_n) = \mathrm{id}_{M_n} = \mu_X^{n,0} \circ M_n(\eta_X)$.

Take $S \colon M_n Q$, for any $Q \in \mathsf{Ob}(\mathscr{E})$. Then consider

$$\begin{aligned} & \mu_Q^{0,n}(\eta_Q(M_n)(S)) \\ =& \mu_Q^{0,n}(\iota_2(\epsilon, S)) \\ =& \{\, \iota_1(\epsilon w) \mid \iota_1(w) \in S \,\} \cup \{\, \iota_2(\epsilon w, q) \mid \iota_2(w, q) \in S \,\} \\ =& S \\ =& \{\, \iota_1(w) \mid \iota_1(w) \in S \,\} \cup \{\, \iota_2(w, q) \mid \iota_2(w, q) \in S \,\} \\ =& \mu_Q^{0,n}\left(\{\, \iota_1(w) \mid \iota_1(w) \in S \,\} \cup \{\, \iota_2(w, \iota_2(\epsilon, q)) \mid \iota_2(w, q) \in S \,\}\right) \\ =& \mu_Q^{0,n}(M_n(\eta_Q)(S)) \end{aligned}$$

■

Proposition 4.2. *The above M_n satisfies the* graded multiplication law*.*

Proof. As before, consider for any $S\colon M_n M_k M_m Q$ the following chain of equations:

$$
\begin{aligned}
&\mu_X^{n,k+m}(M_n(\mu_X^{k,m})(S))\\
=&\mu_X^{n,k+m}\left(\begin{array}{l}\{\,\iota_1(w)\mid\iota_1(w)\in S\,\}\cup\\ \left\{\,\iota_2(w,x)\,\middle|\,\iota_2(w,S')\in S, x\in\mu_X^{k,m}(S')\,\right\}\end{array}\right)\\
=&\mu_X^{n,k+m}\left(\begin{array}{l}\{\,\iota_1(w)\mid\iota_1(w)\in S\,\}\cup\\ \{\,\iota_2(w,\iota_1(w'))\mid\iota_2(w,S')\in S,\iota_1(w')\in S'\,\}\cup\\ \{\,\iota_2(w,\iota_1(w'w''))\mid\iota_2(w,S')\in S,\iota_2(w',q)\in S',\iota_1(w'')\in S''\,\}\cup\\ \{\,\iota_2(w,\iota_2(w'w'',q))\mid\iota_2(w,S')\in S,\iota_2(w',S'')\in S',\iota_2(w'',q)\in S''\,\}\end{array}\right)\\
=&\mu_X^{n,k+m}\left(\begin{array}{l}\{\,\iota_1(w)\mid\iota_1(w)\in S\,\}\cup\\ \{\,\iota_2(w,\iota_1(w'))\mid\iota_2(w,S')\in S,\iota_1(w')\in S'\,\}\cup\\ \{\,\iota_2(w,\iota_1(w'w''))\mid\iota_2(w,S')\in S,\iota_2(w',q)\in S',\iota_1(w'')\in S''\,\}\cup\\ \{\,\iota_2(w,\iota_2(w'w'',q))\mid\iota_2(w,S')\in S,\iota_2(w',S'')\in S',\iota_2(w'',q)\in S''\,\}\end{array}\right)\\
=&\left(\begin{array}{l}\{\,\iota_1(w)\mid\iota_1(w)\in S\,\}\cup\\ \{\,\iota_1(ww')\mid\iota_2(w,S')\in S,\iota_1(w')\in S'\,\}\cup\\ \{\,\iota_1(ww'w'')\mid\iota_2(w,S')\in S,\iota_2(w',S'')\in S',\iota_1(w'')\in S''\,\}\cup\\ \{\,\iota_2(ww'w'',q)\mid\iota_2(w,S')\in S,\iota_2(w',S'')\in S',\iota_2(w'',q)\in S''\,\}\end{array}\right)\\
=&\mu_X^{n+k,m}\left(\begin{array}{l}\{\,\iota_1(w)\mid\iota_1(w)\in S\,\}\cup\\ \{\,\iota_1(ww')\mid\iota_2(w,S')\in S,\iota_1(w')\in S'\,\}\cup\\ \{\,\iota_2(ww',\iota_1(w''))\mid\iota_2(w,S')\in S,\iota_2(w',S'')\in S',\iota_1(w'')\in S''\,\}\cup\\ \{\,\iota_2(ww',\iota_2(w'',q))\mid\iota_2(w,S')\in S,\iota_2(w',S'')\in S',\iota_2(w'',q)\in S''\,\}\end{array}\right)\\
=&\mu_X^{n+k,m}\left(\begin{array}{l}\{\,\iota_1(w)\mid\iota_1(w)\in S\,\}\cup\\ \{\,\iota_1(ww')\mid\iota_2(w,S')\in S,\iota_1(w')\in S'\,\}\cup\\ \{\,\iota_2(ww',\iota_1(w''))\mid\iota_2(w,S')\in S,\iota_2(w',S'')\in S',\iota_1(w'')\in S''\,\}\cup\\ \{\,\iota_2(ww',\iota_2(w'',q))\mid\iota_2(w,S')\in S,\iota_2(w',S'')\in S',\iota_2(w'',q)\in S''\,\}\end{array}\right)\\
=&\mu_X^{n+k,m}\left(\begin{array}{l}\{\,\iota_1(w)\mid\iota_1(w)\in S\,\}\cup\\ \{\,\iota_1(ww')\mid\iota_2(w,S')\in S,\iota_1(w')\in S'\,\}\cup\\ \{\,\iota_2(ww',x)\mid\iota_2(w,S')\in S,\iota_2(w',S'')\in S',x\in S''\,\}\end{array}\right)\\
=&\mu_X^{n+k,m}(\mu_X^{n,k}(M_m)(S)).
\end{aligned}
$$

■

Having established that M_n is a graded monad, the next step is to verify that $\alpha\colon G\Rightarrow M_1$ is a natural transformation.

We can restate the component- and point-wise definition of α from Equation (2.21) in the internal logic of $\mathscr{E}$:

$$
\alpha((o,t))=\{\,\iota_1(s,q)\mid t(s)=q\,\}\cup\begin{cases}\{\iota_2\,\epsilon\} & \text{if } o=\mathrm{true}\\ \{\} & \text{otherwise}\end{cases}
\tag{4.4}
$$

Proposition 4.3. *α is natural.*

Proof. This is to say that

$$
\begin{array}{ccccc}
\Omega\times\mathbf{P}\,(Q)^{\Sigma} & \xrightarrow{\alpha_Q} & \mathbf{P}\,(\Sigma^0+\Sigma^1\times Q) & \xrightarrow{\cong} & \mathbf{P}\,\Sigma^0\times\mathbf{P}\,(Q)^{\Sigma^1}\\
\big\downarrow{\scriptstyle Gf} & & \big\downarrow{\scriptstyle M_1 f} & & \\
\Omega\times\mathbf{P}\,(Q')^{\Sigma} & \xrightarrow{\alpha_{Q'}} & \mathbf{P}\,(\Sigma^0+\Sigma^1\times Q') & \xrightarrow{\cong} & \mathbf{P}\,\Sigma^0\times\mathbf{P}\,(Q')^{\Sigma^1}
\end{array}
\tag{4.5}
$$

commutes.

Given an arbitrary $(o,t) \in GQ$, consider

$$\begin{aligned}
&M_1(f)(\alpha_Q((o,t)))\\
=&M_1(f)\left(\{\,\iota_1(s,q) \mid t(s)=q\,\} \cup \begin{cases}\{\iota_2\,\epsilon\} & \text{if } o = \text{true}\\ \{\} & \text{otherwise}\end{cases}\right)\\
=&\{\,\iota_1(s,f(q)) \mid t(s)=q\,\} \cup \begin{cases}\{\iota_2\,\epsilon\} & \text{if } o = \text{true}\\ \{\} & \text{otherwise}\end{cases}\\
=&\{\,\iota_1(s,q) \mid (ft)(s)=q\,\} \cup \begin{cases}\{\iota_2\,\epsilon\} & \text{if } o = \text{true}\\ \{\} & \text{otherwise}\end{cases}\\
=&(\alpha_{Q'}((o,f\circ t)))\\
=&(\alpha_{Q'}(G(f)((o,t))))
\end{aligned}$$

■

With this result we have verified that all the prerequisites have been met to construct the α-pretrace sequence using the graded trace semantics $\left(\gamma^{(n)}\right)_{n\in\mathbb{N}}$.

4.2 α-pretrace sequence and Accepted Run Semantics

To demonstrate that the α-pretrace sequence semantics and the accepted runs semantics coincide, we demonstrate mutual subsumption.

Theorem 4.4. *Each word that are part of the accepted runs semantics are also part of the α-pretrace semantics.*

First recall Theorem 2.15, allowing us to limit our considerations to single-initial-state q automata, without loss of generality. Furthermore as pointed out in Remark 3.2, we know that any automaton A has a coalgebra represented in $\mathscr{E}$ by the morphism $Q \xrightarrow{\langle o,t\rangle} \Omega \times \mathbf{P}(Q)^{\Sigma}$.

We can take Theorem 4.4 to express, that for any $n \in \mathbb{N}$,

$$\pi_2^n(\mathsf{AccRun}_A^{(n)}) = \left(\pi_{(n+1)} \circ \gamma^{(n+1)}\right)(q) \cap \Sigma^n \subseteq \left(\pi_{(n+1)} \circ \gamma^{(n+1)}\right)(q). \tag{4.6}$$

Proof. We begin with a case-distinction to distinguish between empty and non-empty words. For empty-words and a state q, we consider the following:

$$\begin{aligned}
&\epsilon \in \pi_2^n(\mathsf{AccRun}_A^{(0)})\\
\iff& q \in I \wedge q \in F && \texttt{TODO}\\
\iff& q \in \{q\} \wedge q \in F && \text{By } A \text{ being SISNDA}\\
\iff& q \in F\\
\iff& o(q)\\
\iff& \epsilon \in \{\,\epsilon \mid o(q)\,\}\\
\iff& \epsilon \in \{\,\epsilon \mid \pi_1(\gamma(q))\,\} && \texttt{TODO}\\
\iff& \epsilon \in \left(\left(\pi_1 \circ \gamma^{(1)}\right)(q)\right) \cap \Sigma^0
\end{aligned}$$

Now consider a non-empty word $w = s_1 \ldots s_{n+1}$ and the potentially empty subword $w' =$

$s_2 \dots s_{n+1}$, where the annotated steps are elaborated on below:

$$
\begin{aligned}
& w \in \pi_2^n(\mathsf{AccRun}_A^{(n+1)}) \\
\iff & \exists\, r \colon (Q \times \Sigma \times Q)^{n+1}.\, \pi_{1,1}(r) \in I \wedge \pi_{3,n+1}(r) \in F \wedge \mathsf{Acc}_{(n+1)}(r) \wedge \pi_2^{n+1}(r) = w \\
\overset{(\dagger)}{\iff} & \exists\, r \colon (Q \times \Sigma \times Q)^{n}.\, \pi_{3,n}(r) \in F \wedge \mathsf{Acc}_{(n)}(r) \wedge s_1 \pi_2^n(r) = w \\
\overset{(*)}{\iff} & \exists\, q' \in \gamma(q)(s_1).\, \iota_1(\overbrace{s_2 \dots s_{n+1}}^{w'}) \in \gamma^{(n)}(q') \\
\iff & \exists\, q' \in \gamma(q)(s_1).\, \iota_1(w') \in \gamma^{(n)}(q') \\
\iff & w \in \left(\left(\pi_1 \circ \gamma^{(n+1)}\right)(q)\right) \cap \Sigma^{n+1}
\end{aligned}
$$

Step ($\dagger$) We are making use of the fact of the assumption that there is a single initial state $I = \{q\}$, and therefore fix this in the accepted run. Doing so allows us to focus only on the remaining run, as we know that $\pi_1\left(\mathsf{AccRun}_A^{(n+1)}\right) = (q, s_1, q')$, for some $q' \in Q$.

Step ($*$) **in the "$\Rightarrow$" direction** Given an accepted run, we know there is a sequence of states and input symbols

$$q_1, s_1, q_2, s_2, \dots, q_{n-1}, s_{n-1}, q_n, s_n, q_{n+1}$$

with q_{n+1} being an accepting state. It is clear that $q = q_1$ plays the role of the witness q'. From this state, $\gamma^{(n)}(q')$ contains all words up to a length of n. As the underlying coalgebra corresponds directly to the automaton, we know that the word $s_2 \dots s_{n+1}$ of length $n-1$ must be contained in this subobject.

Step ($*$) **in the "$\Leftarrow$" direction** Given the step from q to q' via s_1, we know that the remaining word $s_2 \dots s_{n+1}$ is accepted. This means that q_{n+1} is in a final state ($q_{n+1} \in F$) and that there is an accepting chain of transitions ($\mathsf{Acc}_{(n)}(r)$). ∎

Let us convince ourselves of the intuitive fact, that the α-pretrace semantics of greater depth contain strictly more words, as each "level" collects *all* words up to a given depth:

Lemma 4.5. *All accepted words in $\gamma^{(n)}(q)$ are contained in $\gamma^{(n+1)}(q)$.*

Proof. Consider the base-case $n = 0$,

$$
\begin{aligned}
& \left(\pi_{(0)} \circ \gamma^{(0)}\right)(q) \\
= & \{\, \epsilon \mid \pi_1(\gamma(q)) \,\} \\
\subseteq & \{\, \epsilon \mid \pi_1(\gamma(q)) \,\} \cup \{\, \langle s \rangle \mid \exists\, q' \in \gamma(q)(s).\, \pi_1(\gamma(q')) \,\} \\
= & \{\, \epsilon \mid \pi_1(\gamma(q)) \,\} \cup \left\{\, sw \;\middle|\; \exists\, q' \in \gamma(q)(s).\, \iota_1 w \in \gamma^{(0)}(q') \,\right\} \\
= & \left(\pi_{(1)} \circ \gamma^{(1)}\right)(q).
\end{aligned}
$$

As for the induction-step, grant $\left(\pi_{(n-1)} \circ \gamma^{(n-1)}\right)(q) \subseteq \left(\pi_{(n)} \circ \gamma^{(n)}\right)(q)$ for $n \geq 1$:

$$
\begin{aligned}
& \left(\pi_{(n)} \circ \gamma^{(n)}\right)(q) \\
= & \{\, \epsilon \mid \pi_1(\gamma(q)) \,\} \cup \left\{\, sw \;\middle|\; \exists\, q' \in \gamma(q)(s).\, \iota_1 w \in \gamma^{(n-1)}(q') \,\right\} \\
\subseteq & \{\, \epsilon \mid \pi_1(\gamma(q)) \,\} \cup \left\{\, sw \;\middle|\; \exists\, q' \in \gamma(q)(s).\, \iota_1 w \in \gamma^{(n)}(q') \,\right\} && \text{By I.H.} \\
= & \left(\pi_{(n+1)} \circ \gamma^{(n+1)}\right)(q).
\end{aligned}
$$

∎

Lemma 4.6. *Each level $n+1$ of the α-pretrace semantics* only *adds words of length n.*

Proof. Expressed symbolically, we want to prove $(\pi_{(n+1)} \circ \gamma^{(n+1)}(q)) \setminus (\pi_{(n)} \circ \gamma^{(n)}(q)) \subseteq \Sigma^n$ for any state q, by induction:

Induction Basis For $n = 0$, we consider that

$$\left(\left\{ sw \;\middle|\; \exists\, q' \in \gamma(q)(s).\, \iota_1(\epsilon) \in \gamma^{(0)}(q') \right\} \cup \{\, \epsilon \mid \pi_1(\gamma(q)) \,\}\right) \setminus \{\, \epsilon \mid \pi_1(\gamma(q)) \,\} \subseteq \Sigma^1$$

holds as it is easy to see, $\{\, \epsilon \mid \pi_1(\gamma(q)) \,\}$ is removed from $\gamma^{(1)}(q)$, leaving us only with singleton words.

Induction Step Setting aside the empty word, it is easy to see that

$$\begin{aligned}
&(\pi_{(1)} \circ \gamma^{(n+1)}(q)) \setminus (\pi_{(1)} \circ \gamma^{(n)}(q)) \\
=&\left\{ sw \;\middle|\; \exists\, q' \in \gamma(q)(s).\, \iota_1(w) \in \gamma^{(n)}(q') \right\} \setminus \left\{ sw \;\middle|\; \exists\, q' \in \gamma(q)(s).\, \iota_1(w) \in \gamma^{(n-1)}(q') \right\} \\
=&\left\{ sw \;\middle|\; \exists\, q' \in \gamma(q)(s).\, \iota_1(w) \in \left(\gamma^{(n)}(q') \setminus \gamma^{(n-1)}(q')\right) \right\} \subseteq \Sigma^{n+1}
\end{aligned}$$

must hold, as by the induction hypothesis $\gamma^{(n)}(q') \setminus \gamma^{(n-1)}(q')$ consists only of words of length n, and hence the extension by a input symbol s results in words of only length $n+1$. ■

We can restate this result as saying $\gamma^{(n+1)} \setminus \gamma^{(n)} = \gamma^{(n+1)} \cap \Sigma^n$.

Theorem 4.7. *The α-pretrace semantics only contains words contained in the accepted runs semantics.*

In this case we consider the union of the accepted runs up to a length of n, and even strengthen the statement:

$$\bigcup_{i=0}^{n} \pi_2^i(\mathsf{AccRun}_A^{(i)}) = \left(\pi_{(n+1)} \circ \gamma^{(n+1)}\right)(q). \tag{4.7}$$

Proof. We proceed by induction over n. In the base-case $n = 0$, we have the same situation as in the proof of Theorem 4.4.

From Lemma 4.5, we can derive that the difference

$$\left(\pi_{(n+1)} \circ \gamma^{(n+1)}\right)(q) \subseteq \left(\pi_{(n)} \circ \gamma^{(n)}\right)(q)$$

denotes all *new* words recognised by the next depth. As such, we know that all words in the difference must be of length n:

$$= \left(\pi_{(n+1)} \circ \gamma^{(n+1)}\right)(q) \cap \Sigma^{n+1}.$$

For the induction step, we assume that Equation (4.7) holds for $n-1$.

$$\begin{aligned}
&\bigcup_{i=0}^{n} \pi_2^n(\mathsf{AccRun}_A^{(i)}) = \left(\pi_{(n+1)} \circ \gamma^{(n+1)}\right)(q) \\
\iff &\bigcup_{i=0}^{n} \pi_2^i(\mathsf{AccRun}_A^{(i)}) \setminus \bigcup_{i=0}^{n-1} \pi_2^i(\mathsf{AccRun}_A^{(i)}) = \left(\pi_{(n+1)} \circ \gamma^{(n+1)}\right)(q) \setminus \left(\pi_{(n)} \circ \gamma^{(n)}\right)(q) \\
\overset{(*)}{\iff} &\pi_2^n(\mathsf{AccRun}_A^{(n)}) = \left(\pi_{(n+1)} \circ \gamma^{(n+1)}\right)(q) \cap \Sigma^{n+1}.
\end{aligned}$$

In the last inference $(*)$ uses Lemma 4.6 to reshape the right-hand-side of the equation, and holds by Theorem 4.4. ■

These results give us an alternative perspective on categorical automata in an arbitrary topos $\mathscr{E}$, i.e. without having to assume that $\mathscr{E}$ is countably extensible.

5 Summary

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